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Title : Construction of DFAs

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This class deals with the different constuction, or operations that can be performed on deterministic finite accepters (DFAs). Recall that a DFA is defined as a 5-tuple

$$M = (Q, \Sigma, q_0, \delta, F)$$

where Q is a set of states, Σ is the defined alphabet, q_0 is the starting state, Δ is the transition function between states, and F is the set of final states. The constructions on DFAs to be looked at here are union, intersection and complement.

1 Complement

Given a DFA

$$M1 = (Q_1, \Sigma, p_0, \delta_1, F_1),$$

we seek to construct aother DFA

$$M = (Q_1, \Sigma, p_0, \delta_1, Q_1 - F_1),$$

so that the following equation holds:

$$L(M_1) = \overline{L(M)}$$

This means the language accepted by M is the complement of the language accepted by M_1; in other words, every input string that is accepted by M_1 is rejected by M, and vice versa.

Consider the tuple notation for each DFA: they have the same set of states, alphabet, start state and transition function, but the set of final states for M is the opposite of those for M_1 (hence it is the set $Q_1 - F_1$, all states in Q_1 that are not in F_1).

Proof: There are two parts for the proof, corresponding to what is required by set equality.

(1): $L(M) \subseteq \overline{L(M_1)}$. I.e., the language accepted by M is a subset of the complement of the language accepted by M_1.

For any w in $L(M)$,

$$p_0 \xrightarrow{w}{}^{*}_{M} p,$$

where p is in the set $Q_1 - F_1$.

Therefore $p \notin F_1$, i.e. $w \notin L(M_1)$. So $w \in \overline{L(M_1)}$.

(2): $\overline{L(M_1)} \subseteq L(M)$. I.e., the complement of the language accepted by M_1 is a subset of the language accepted by M. (This can be shown in a manner similar to the above method)

And since the 2 items are subsets of each other, they are shown to be equivalent.

2 Intersection

The intersection $\cap$ of languages $L(M_1)$ and $L(M_2)$, where

$$M_1 = (Q_1, \Sigma, p_0, \delta_1, F_1)$$

and

$$M_2 = (Q_2, \Sigma, q_0, \delta_2, F_2)$$

is $L(M)$, where

$$M = (Q_1 \times Q_2, \Sigma, (p_0, q_0), \delta, F).$$

Here, M represents the product of each dfa's set of states, the combined start state of each dfa, and the product of each dfa's set of final states ($F = F_1 \times F_2$), where the dfas share the same aplhabet and the transition state works as follows:

$$\delta((p,q),a) = (p',q') \Leftrightarrow \delta(p,a) = p' \ \&\ \delta(q,a) = q'.$$

We prove that M meets its specification. String $a_1 a_2 \cdots a_n$ is in $L(M_1) \cap L(M_2)$ iff

$$p_0 \longrightarrow^{a_1} r_1 \longrightarrow^{a_2} r_2 \longrightarrow \ldots \longrightarrow^{a_n} r_n,$$

where $r_n \in F_1$ and also

$$q_0 \longrightarrow^{a_1} s_1 \longrightarrow^{a_2} s_2 \ldots \longrightarrow^{a_n} s_n,$$

where $s_n \in F_2$; This is the case iff

$$(p_0, q_0) \longrightarrow^{a_1} (r_1, s_1) \longrightarrow^{a_2} (r_2, s_2) \ldots \longrightarrow^{a_n} (r_n, s_n),$$

and $(r_n, s_n) \in F$.

This, in turn, is true iff $a_1 a_2 \cdots a_n \in L(M)$.

3 Union

The union $\cup$ of languages $L(M_1)$ and $L(M_2)$, where

$$M_1 = (Q_1, \Sigma, p_0, \delta_1, F_1),$$

and

$$M_2 = (Q_2, \Sigma, p_0, \delta_2, F_2)$$

is $L(M)$, where

$$M = (Q_1 \times Q_2, \Sigma, (p_0, q_0), \delta, F)$$

with $F = (F_1 \times Q_2) \cup (Q_1 \times F_2)$, and δ is defined the same as the case for intersection.

We show that $L(M) = L(M_1) \cup L(M_2)$.

Proof:

$$\begin{aligned} a_1 \ldots a_n \in L(M_1) & \Rightarrow p_0 \longrightarrow^{a_1} r_1 \cdots \longrightarrow^{a_n} r_n \;\&\; r_n \in F_1 \\ & \Rightarrow (p_0, q_0) \longrightarrow^{a_1} (r_1, s_1) \cdots \longrightarrow^{a_n} (r_n, s_n) \;\&\; (r_n, s_n) \in F_1 \times Q_2 \\ & \Rightarrow a_1 \cdots a_n \in L(M) \end{aligned}$$

This shows $L(M_1) \subseteq L(M)$. Similarly, we can show $L(M_2) \subseteq L(M)$, and so

$$L(M_1) \cup L(M_2) \subseteq L(M).$$

On the other hand,

$$\begin{aligned} a_1 \dots a_n \in L(M_1) \ &\Rightarrow (p_0, q_0) \longrightarrow^{a_1} (r_1, s_1) \dots \longrightarrow^{a_n} (r_n, s_n) \\ &\quad \& \ (r_n, s_n) \in (F_1 \times Q_2) \cup (Q_1 \times F_2) \end{aligned}$$

Either $(r_n, s_n) \in (F_1 \times Q_2)$, or $(r_n, s_n) \cup (Q_1 \times F_2)$. If $(r_n, s_n) \in (F_1 \times Q_2)$, then we have $a_1 a_2 \cdots a_n \in L(M_1)$. If $(r_n, s_n) \in (Q_1 \times F_2)$, then we have $a_1 a_2 \cdots a_n \in L(M_2)$. In either case, we have $a_1 a_2 \cdots a_n \in L(M_1) \cup L(M_2)$.