CS865 Homework Solutions

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Exercise 1.1 Prove that every non-empty set of natural numbers has a least element.

Proof

1. Let's assume by reductio that X has no least element.
2. We prove, by strong induction, that X must be empty, contradicting the assumption that X is non-empty. Specifically, we show that the property

   $$P(k) :\equiv \ k \notin X$$

   holds for all $k \in \mathbb{N}$, so X cannot contain any natural number.
3. Base case: $P(0)$ is true because 0 cannot be a member of X, for otherwise 0 would be the least element, contradicting ①.
4. Induction step: Let's now assume by strong induction that, given an arbitrary $n \in \mathbb{N}$, $i \notin X$ holds for all $i < n$.
5. Suppose that $n \in X$; then, by ④, n is the least element of X, which contradicts ①. Therefore: $n \in \mathbb{N} - \mathrm{X}$, which concludes the inductive step.
6. This establishes our goal set in ②.

Exercise 1.2 Show that if the property E of natural numbers and $p \in \mathbb{N}$ are such that (i) $E(p)$; and (ii) $\forall n \in \mathbb{N}\,[\mathrm{E(n)} \Rightarrow \mathrm{E(n+1)}]$, then $\forall n \geq p\ E(n)$.

Proof Let's abbreviate $E(p+n)$ as $F(n)$. We prove that $F(n)$ holds for all $n \geq 0$, which is equivalent to saying that $E(p+n)$ holds for all $n \geq 0$, or $E(m)$ holds for all $m \geq p$.

1. Base case: It is given that $E(p)$ holds, for some $p \in \mathbb{N}$. So, $F(0)$ holds, because $F(0)$ is $E(p+0)$.

2. Induction step: It is also given that form E, we have

$$\forall n \in \mathbb{N}\,[\mathrm{E(n)} \Rightarrow \mathrm{E(n+1)}].$$

In particular, we have

$$\forall n \in \mathbb{N}\,[\mathrm{E(p+n)} \Rightarrow \mathrm{E(p+(n+1))}];$$

but this is the same as saying that if $F(n)$ holds, then so does $F(n+1)$, as required by the induction step.

3. By the principle of mathematical induction, we have shown that $\forall n \geq 0$, $F(n)$ holds.

Exercise 1.3 Assume that $E(0)$, $E(1)$, and $E(n) \Rightarrow E(n+2)$.
Show that $\forall n \in \mathbb{N}\ \ E(n)$.

Proof

Let $F(n)$ stands for the property $E(2n) \wedge E(2n+1)$; that is, $F(n)$ is true exactly when both $E(2n)$ and $E(2n+1)$ are true. If we can show that $F(n)$ is true for all $n \geq 0$, then we have achieved our goal because this means $E(n)$ holds for all even numbers and all odd numbers.

Base case : $F(0)$ holds, by the assumption that both $E(0)$ and $E(1)$ holds.

Induction step : Now suppose $F(k)$ holds for some $k \geq 0$. This means both $E(2k)$ and $E(2k+1)$ are assumed true. We want to show that $F(k+1)$ is true, that is, both $E(2k+2)$ and $E(2k+3)$ are true. However, we are given that

$$\forall n \geq 0\ E(n) \Rightarrow E(n+2).$$

So from this and the assumption that $E(2k)$ holds we know $E(2k+2)$ holds. Similarly, from the assumption that $E(2k+1)$ holds we know $E(2k+3)$ holds. Therefore, $F(k+1)$ is true.

Exercise 1.4 Assume the function $f : \mathbb{N} \to \mathbb{N}$ is such that $\forall n \in \mathbb{N}\ (\mathrm{f(n+1)} \leq \mathrm{f(n)})$.

1. Show that for some $m \in \mathbb{N}\ :\ \mathrm{f(m+1) = f(m)}$.

2. Show that for some $m \in \mathbb{N}\ :\ \forall \mathrm{n}\ \ \mathrm{f(m+n) = f(m)}$.

Proof

1A Proof Using Exercise 1.1

1. Consider the set $f^{\star}(\mathbb{N}) :\equiv \{\mathrm{f(k)} \mid \mathrm{k} \geq 0\}$, which is usually called the image of $\mathbb{N}$ under f. Since f is a function and $\mathbb{N}$ is not empty, $f^{\star}(\mathbb{N})$ must be non-empty as well.

2. By Exercise 1.1, every non-empty set of natural numbers has a least member. So the set $f^{\star}(\mathbb{N})$ also has a least element, which can be written as $f(m)$ for some $m \in \mathbb{N}$. (Note that all members in $f^{\star}(\mathbb{N})$ are the result of applying f to some natural number.)
3. For this particular m in ②, the given condition instanciates to $f(m+1) \leq f(m)$. But ② concludes $f(m) \leq f(m+1)$, since $f(m)$ is the least one. Therefore, $f(m+1) = f(m)$, as required.

1B Alternative Proof

1. Assume, by reductio, that for no m do we have $f(m+1) = f(m)$. By the assumption that $f(n+1) \leq f(n)$ for all $n \geq 0$, we have $f(n+1) < f(n)$ for all $n \geq 0$.
2. We prove, by mathematical induction, that $f(0) \geq m$ for any $m \geq 0$, which is clearly impossible.
3. Base case $f(0) \geq 0$: this is true since f is a function from $\mathbb{N}$ to $\mathbb{N}$.
4. Induction step: Assume that $f(0) \geq k$ for some $k \in \mathbb{N}$. We show that $f(0) \geq k+1$ also holds. In fact,

$$\begin{aligned} f(0) &= [f(0) - f(1)] + [f(1) - f(2)] + \cdots + [f(k) - f(k+1)] + f(k+1) \\ &\geq 1 + 1 + 1 + 1 + \cdots + 1 + f(k+1) \\ &\geq (k+1) + f(k+1) \\ &\geq k+1. \end{aligned}$$

   Thus $f(0) \geq k+1$. This is what needs to be proved. Note that here we did not really need to use the induction hypothesis.

2 This is similar to 1, and we use the result of Exercise 1.1.

1. Consider the set $f^{\star}(\mathbb{N}) :\equiv \{\mathrm{f(k)} \mid \mathrm{k} \geq 0\}$, which is usually called the image of $\mathbb{N}$ under f. Since f is a function and $\mathbb{N}$ is not empty, $f^{\star}(\mathbb{N})$ must be non-empty as well.
2. By Exercise 1.1, every non-empty set of natural numbers has a least member. So the set $f^{\star}(\mathbb{N})$ also has a least element, which can be written as $f(m)$ for some $m \in \mathbb{N}$. (Note that all members in $f^{\star}(\mathbb{N})$ are the result of applying f to some natural number.)
3. For this particular m in ②, the given condition instanciates to $f(m+n) \leq f(m)$ for any $n \geq 0$. (Strictly speaking, an easy proof by induction is involved here.) But ② concludes $f(m) \leq f(k)$ for any $k \geq 0$, since $f(m)$ is the least one. Therefore, $f(m+n) = f(m)$ for all $n \geq 0$, as required.

Exercise 1.5 Assume that $f: \mathbb{N} \rightarrow \mathbb{N}$ is such that

$$\forall n, m \in \mathbb{N} \ (\mathrm{n} < \mathrm{m} \Rightarrow \mathrm{f(n)} < \mathrm{f(m)}).$$

Show:

$$\forall n \in \mathbb{N} \ \mathrm{n} \leq \mathrm{f(n)}.$$

Proof

1. Inductive basis: $0 \leq f(0)$ holds because f is a function from $\mathbb{N}$ to $\mathbb{N}$ and 0 is the least element of $\mathbb{N}$.
2. Induction hypothesis: Assume that $n \leq f(n)$ holds for some $n \in \mathbb{N}$.
3. Inductive step: We need to prove that $n+1 \leq f(n+1)$. By the given assumption, we have $f(n+1) > f(n)$. The induction hyppothesis gives us $f(n) \geq n$. So, $f(n+1) > n$ and hence, $f(n+1) \geq n+1$.

Exercise 2.7 Show that the process of repeatedly applying steps 1–4 in Theorem 2.9 (Conjunctive Normal Form) (i) terminates, and (ii) produces, in fact, a formula in conjunctive normal form.

Proof

Given a formula ϕ_0, we can assume ϕ_0 contains no such subformulas as $\phi \to \psi$ and $\phi \leftrightarrow \psi$, since if ϕ_0 does, ϕ_0 has finite such subformulas. We replace $\phi \to \psi$ and $\phi \leftrightarrow \psi$ by $\neg\phi \vee \psi$ and $(\neg\phi \vee \psi) \wedge (\phi \vee \neg\psi)$, resp.

1. Denote

$$\mathcal{M}_{\phi_0} = \max\{k \mid \neg(\phi_1 \wedge \cdots \wedge \phi_k) \text{ or } \neg(\phi_1 \vee \cdots \vee \phi_k) \text{ is a subformula of } \phi_0\}$$

   Since ϕ_0 has finite literals, $\mathcal{M}_{\phi_0} < +\infty$. By replacing subformulas as follows:

$$\begin{array}{lll} \neg(\phi_1 \wedge \cdots \wedge \phi_{\mathcal{M}_{\phi_0}}) & \text{with} & \neg\phi_1 \vee \cdots \vee \neg\phi_{\mathcal{M}_{\phi_0}} \\ \neg(\phi_1 \vee \cdots \vee \phi_{\mathcal{M}_{\phi_0}}) & \text{with} & \neg\phi_1 \wedge \cdots \wedge \neg\phi_{\mathcal{M}_{\phi_0}} \end{array}$$

   and erasing double negations whenever encountered, we can get ϕ_0' such that $\phi_0' \sim \phi_0$ and $\mathcal{M}_{\phi_0'} < \mathcal{M}_{\phi_0}$.

   By using strong mathematical induction on $\mathcal{M}_{\phi}$, we can prove that there is ϕ_{01} such that $\phi_{01} \sim \phi_0$ and $\mathcal{M}_{\phi_{01}} = 1$; i.e., ϕ_{01} has no such subformulas as $\neg(\phi_1 \vee \cdots \vee \phi_k)$ or $\neg(\phi_1 \wedge \cdots \wedge \phi_k)$ where $k \geq 2$.

2. Since ϕ_0 has finite literals and finitely many processes have been applied on ϕ_0 so far, ϕ_{01} has finitely many parentheses in conjunctions and disjunctions. So we erase them to get ϕ_{02} such that $\phi_{02} \sim \phi_{01} \sim \phi_0$.

3. Denote

$$\mathcal{N}_{\phi_{02}} = \max\{k \mid \phi \vee (\psi_1 \wedge \cdots \wedge \psi_k) \text{ or } \phi \wedge (\psi_1 \vee \cdots \vee \psi_k) \text{ is a subformula of } \phi_{02}\}$$

   For the same reason as in (1), $\mathcal{N}_{\phi_0} < +\infty$.

   Using a similar idea as in (1) and using distribution, we can get ϕ_{03} such that $\phi_{03} \sim \phi_{02} \sim \phi_0$ and $\mathcal{N}_{\phi_{03}} = 1$; i.e., ϕ_{03} has no such subformulas as $\phi \vee (\psi_1 \wedge \cdots \wedge \psi_k)$ or $\phi \wedge (\psi_1 \vee \cdots \vee \psi_k)$ where $k \geq 2$.

4. If necessary, apply (2) again. We can get ϕ_{04} such that $\phi_{04} \sim \phi_{03} \sim \phi_0$ and ϕ_{04} is in CNF.

   By now, we have proved that the process in the proof of Theorem 2.9 (i) terminates, and (ii) produces a formula in CNF, as required.

Exercise 2.9

Thesis For any fromula ϕ containing only the connectives $\neg, \wedge, \vee$,

$$\models \phi \text{ iff } \models \neg\phi^d$$

where ϕ^d is the formula abtained from ϕ by exchanging "$\vee$" with "$\wedge$" and vice-versa.

Preliminary Note

If $\mathcal{S}$ is the set of all well-formed $(\neg, \wedge, \vee)$ formulas, let d:$\mathcal{S} \to \mathcal{S}$ be defined as follows:

$$d(\phi) = \phi^d$$

Then d is an automorphism of $\mathcal{S}$ with respect to the relation of equivalence $\sim$ in $\mathcal{S}$, and we have:

$$\forall \phi, \phi' \in \mathcal{S} \tag{1}$$
$$d(\phi) = d(\phi') \Leftrightarrow \phi = \phi' \text{ and} \tag{2}$$
$$\phi \sim \phi' \Leftrightarrow d(\phi) \sim d(\phi') \tag{3}$$

Furthermore, d is such that $d^{-1} = d$, i.e.:

$$\forall \phi \in \mathcal{S} \quad d(d(\phi)) = \phi$$

What precedes can be shown by induction on the maximum length of pairs ϕ and ϕ', but we don't give the formal proof here for sake of brevity.

Demonstration

Let ϕ be an arbitrary formula of $\mathcal{S}$ and let ϕ' be a formula s.t. ϕ' is in CNF and $\phi \sim \phi'$. ϕ' exists, therefore we can always right:

$$\phi \sim \bigwedge_1^m i \bigvee_1^{k_i} j L_{ij} = \phi'$$

where $1 \leq m < \infty$, $1 \leq k_i < \infty$ and L_{ij} represents a literal.

Recalling the preliminary note, we get:

$$d(\phi) \;\sim\; d(\bigwedge_1^m i \bigvee_1^{k_i} j L_{ij}) \tag{4}$$
$$\phi^d \;\sim\; \bigvee_1^m i \bigwedge_1^{k_i} j L_{ij} \tag{5}$$
$$\neg\phi^d \;\sim\; \bigwedge_1^m i \bigvee_1^{k_i} j L'_{ij} \tag{6}$$

Where $L'_{ij} = \neg L_{ij}$; (5) follows from (4) by the definition of d and (6) follows from (5) by the generalized DeMorgan's law.

Left to Right
Assume that $\models \phi$; then, necessarily, $\forall i\, k_i \geq 2$, and every clause

$$\bigvee_{1}^{k_i} j\ L_{ij} \text{ in } \phi$$

must contain at least one pair of complementary literals (this is a property of logically valid clausal forms).
Since $L'_{ij} = \neg L_{ij}$, we see that every such complementary pair must also appear in each correspinding clause in the CNF at (6).
This is sufficient to conclude that such a CNF is logically valid, i.e. $\models \neg\phi^d$

Right to Left
If we recall from the preliminary note the fact that $d = d^{-1}$, we see that we can reason as above to go from (6) to ϕ, and conclude $\models \phi$ under the assumption $\models \neg\phi^d$.

Exercise 2.10 A, B, C, D and E are variables. Give equivalents in conjunctive normal form for:

1. $(A \wedge B) \leftrightarrow (C \vee D)$
   $[(A \wedge B) \leftrightarrow (C \vee D)] \sim [(\neg A \vee \neg B \vee C \vee D) \wedge (A \vee \neg C) \wedge (A \vee \neg D) \wedge (B \vee \neg C) \wedge (B \vee \neg D)]$

2. $(((\neg A \vee B) \wedge \neg C) \vee D) \wedge \neg E$
   $[(((\neg A \vee B) \wedge \neg C) \vee D) \wedge \neg E] \sim [(\neg A \vee B \vee D) \wedge (\neg C \vee D) \wedge \neg E]$

Exercise 2.15 Being first in conjunctive normal (if necessary) and rewrite clausal (A, B, C are variable):

1. $\neg A$
   CNF: $\neg A$
   CF: $\{\{\neg A\}\}$

2. $A \wedge \neg B$
   CNF: $(A \wedge \neg B)$
   CF: $\{\{A\}, \{\neg B\}\}$

3. $\neg A \vee B$
   CNF: $(\neg A \vee B)$
   CF: $\{\{\neg A, B\}\}$

4. $\neg(\neg A \wedge B \wedge \neg C)$
   CNF: $(A \vee \neg B \vee C)$
   CF: $\{\{A, \neg B, C\}\}$

5. $(\neg A \wedge \neg B \wedge C) \vee (A \wedge \neg B \wedge \neg C)$
   CNF: $((A \vee \neg A) \wedge (A \vee \neg B) \wedge (A \vee C) \wedge (\neg B \vee \neg A) \wedge \neg B \wedge (\neg B \vee C) \wedge (\neg A \vee \neg C) \wedge (\neg B \vee \neg C) \wedge (\neg C \vee C))$
   CF: $\{\{A, \neg A\}, \{A, \neg B\}, \{A, C\}, \{\neg B, \neg A\}, \{\neg B\}, \{\neg B, C\},$
   $\{\neg A, \neg C\}, \{\neg B, \neg C\}, \{\neg C, C\}\}$

6. $A \leftrightarrow (\neg B \wedge C)$
   CNF: $((\neg A \vee B \vee C) \wedge (A \vee \neg B) \wedge (A \vee \neg C))$
   CF: $\{\{\neg A, B, C\}, \{A, \neg B\}, \{A, \neg C\}\}$

Exercise 2.16 Form all possible resolvants of the following pairs of clauses:

1. $\{A, B\}; \{\neg A, \neg B\}$
   $\{A, \neg A\}, \{B, \neg B\}$

2. $\{\neg A, \neg B, \neg C\}; \{\neg B, C\}$
   $\{\neg A, \neg B\}$

3. $\{A, \neg A\}; \{A, \neg A\}$
   $\{A, \neg A\}$

Exercise 2.17 Produce all possible resolvents of two clauses in the following sets

1. $\{\{\neg A, \neg B\}, \{\neg A, B\}, \{A\}\}$
   $\{\neg A\}, \{B\}, \{\neg B\}$

2. $\{\{A, B, C\}, \{\neg B, \neg C\}, \{\neg A, \neg C\}\}$
   $\{A, C, \neg C\}, \{A, B, \neg B\}, \{B, C, \neg C\}, \{A, \neg A, B\}$

3. $\{\{\neg A, \neg B\}, \{B, C\}, \{\neg C, A\}\}$
   $\{\neg A, C\}, \{\neg B, \neg C\}, \{A, B\}$

4. $\{\{\neg A, \neg B\}, \{B, C\}, \{\neg C, A\}\}$
   $\{B, \neg C\}, \{B, C, \neg C\}, \{A, \neg A, B\}$

5. $\{\{\neg A\}, \{A, B, \neg C\}, \{C, \neg A\}\}$
   $\{B, \neg C\}, \{B, C, \neg C\}, \{A, \neg A, B\}$

Exercise 2.18 Show: if E is a resolvent of C and D, and $C = D$, then $E = C = D$.

1. There is some $A \in C$ such that $\neg A \in D$ by the definition of resolvent.

2. $E := C - \{A\} \cup D - \{\neg A\}$

3. But since, $C = D$, $\neg A \in C$ and $A \in D$

4. $\neg A \in C - \{A\}$ and $A \in D - \{\neg A\}$

5. Thus, $C - \{A\} \cup D - \{\neg A\} = C \cup D = C = C$

6. So, $C = D = E$.

Exercise 2.19 Show: if E is a resolvant of C and $\{A, \neg A\}$, then $E = C$.
Either $A \in C$ or $\neg A \in C$
Cases:

1. $E = C - \{A\} \cup \{A, \neg A\} - \{\neg A\}$
   $E = C - \{A\} \cup \{A\} = C$

2. $E = C - \{\neg A\} \cup \{A, \neg A\} - \{A\}$
   $E = C - \{\neg A\} \cup \{\neg A\} = C$

Thus, $E = C$.

Exercise 2.22 Show that every collection with a finite meet has a minimal meet.

<u>Proof</u>

1. Suppose $P = \{A_\alpha \mid \alpha \in \bigwedge\}$ is a collection of sets with a finite meet denoted by
$$J = \{x_i \mid i = 1, 2, ..., n\},$$
where $n < +\infty$.
2. Denote $J_0 = J$ and define
$$J_i = \begin{cases} J_{i-1}, & \text{if } J_{i-1} - \{x_i\} \text{ is not a meet of } P, \\ J_{i-1} - \{x_i\}, & \text{if } J_{i-1} - \{x_i\} \text{ is a meet of } P. \end{cases}$$
Here, $i = 1, 2, ..., n$.
3. So, $J_i (i = 1, 2, ..., n)$ is not a meet of P and $J_0 \supseteq J_1 \supseteq J_2 \supseteq ... \supseteq J_n$.
4. In particular, J_n is a meet of P.
5. For any $x \in J_n$, denote the index of x by i, i.e. $x = x_i$, if $J_n - \{x\}$ is a meet of P, then since $J_{i-1} \supseteq J_n$, $J_{i-1} - \{x\}$ is also a meet of P.
6. By the construction of J_i, $x = x_i \notin J_i$.
7. But $J_i \supseteq J_n$. So $x \notin J_n$, which contradicts $x \in J_n$.
8. Thus $J_n - \{x\}$ is not a meet of P.
9. So J_n is a minimal meet of P.

Exercise 2.23 Prove that every finite collection of non-empty sets has a minimal meet.

<u>Proof</u>

1. Since the collection is finite, we can denote it by $P = \{A_i \mid i = 1, 2, ..., n\}$.
2. For any $i (i = 1, 2, ..., n)$, pick $x_i \in A_i$, and denote $J = \{x_i \mid i = 1, 2, ..., n\}$.
3. Then J is a finite meet of P.
4. By exercise 2.22, P has a minimal meet.

Exercise 2.24 (i) What is the minimal meet for the collection $P := \{\{n, n+1\} \mid n \in \mathbb{N}\}$ produced by the proof of Lemma 2.21?

Answer: The minimal meet is $J = \{2n+1 \mid n \in \mathbb{N}\}$.

Exercise 2.24 (ii) Give the first 14 elements of the minimal meet for $\{\{n, 2n\} \mid n \in \mathbb{N}\}$, produced by this proof.

Answer: The first 14 elements are: 0,2,6,8,10,14,18,22,24,26,30,32,34,38.

Exercise 2.24 (iii) Idem, for $\{\{n, n+1, ..., 2n\} \mid n \in \mathbb{N}\}$, but only the first 6 elements.

Answer: The first 6 elements are: 0,2,6,14,30,62.

Exercise 2.26 A,B, and C are propositional variables. Is $\square$ derivable from the following sets of clauses? If so, give a derivation. If not, give an argument showing that such a derivation does not exist.

(i) $\{\{A, B\}, \{B, C\}, \{A, C\}, \{\neg A, \neg B\}, \{\neg B, \neg C\}, \{\neg A, \neg C\}\}$.

Answer: Yes, $\square$ is derivable, and here is a derivation.

1. $\{A, C\}, \{\neg A, \neg B\} \rightarrow \{C, \neg B\}$
2. $\{A, B\}, \{\neg A, \neg C\} \rightarrow \{B, \neg C\}$
3. $\{C, \neg B\}, \{B, C\} \rightarrow \{C\}$
4. $\{B, \neg C\}, \{\neg B, \neg C\} \rightarrow \{\neg C\}$
5. $\{C\}, \{\neg C\} \rightarrow \square$

(ii) $\{\{A, B\}, \{B, C\}, \{\neg A, \neg B\}, \{\neg B, \neg C\}, \{\neg A, \neg C\}\}$.

Answer: No, and here is why. We give a truth assignment γ as follows:
$t \rightarrow B, f \rightarrow A, f \rightarrow C$, then, clearly $\gamma \models \{\{A, B\}, \{B, C\}, \{\neg A, \neg B\}, \{\neg B, \neg C\}, \{\neg A, \neg C\}\}$.

By Lemma Corollary 2.17, $\square$ is not derivable.

Exercise 2.28 Give a simple example showing that $\Sigma \models C \Rightarrow \Sigma \vdash_p C$ can be false when $C \neq \square$.

e.g. $\Sigma = \{\{A, B\}, \{\neg A, B\}\}, C = \{B, D\}$, where $D \neq A, B, \neg A, \neg B$.

Clearly $\Sigma \not\vdash_p C$, but $\Sigma \models C$.

Exercise 2.35 Prove Lemma 2.21 without assuming that $\bigcup_P$ is countable.

<u>Proof</u>

1. Suppose $P = \{A_\alpha \mid \alpha \in \bigwedge\}$ is a collection of finite, non-empty sets.
2. Denote $Q = \{B_\beta \mid \beta \in \bigwedge'$ and B_β is a meet of $P\}$.
3. Since
$$\bigcup_P = \bigcup_{\alpha \in \bigwedge} A_\alpha$$
is a meet of P, Q is nonempty.
4. Under the relation of set containment, Q is a poset.
5. Suppose $O = \{C_\alpha \mid \alpha \in \bigwedge''\} \subseteq Q$ is a chain.
6. Denote
$$W = \bigcap_{\alpha \in \bigwedge''} C_\alpha$$
We claim that $W \in Q$ is the least element of O.
7. Prove $W \in Q$, i.e. W is a meet of P: Given any $A \in P$, we can denote $A = \{x_1, x_2, ..., x_n\}$ since A is finite and nonempty.
8. We want to show $W \cap A \neq \emptyset$.
9. Suppose not, i.e. $W \cap A = \emptyset$.
$$\Big(\bigcap_{\alpha \in \bigwedge''} C_\alpha\Big) \cap A = \emptyset$$
10. Then for any $x_i \in A (i = 1, 2, ..., n)$, there exists $\alpha_i \in \bigwedge''$, such that $x_i \notin C_{\alpha_i}$.
11. Since O is a chain, without loss of generalization, we assume $C_{\alpha_1} \subseteq C_{\alpha_2} \subseteq ... \subseteq C_{\alpha_n}$.
12. So $x_i \notin C_{\alpha_i}$ for $i = 1, 2, ..., n$.
13. So $A \cap C_{\alpha_1} = \emptyset$, which contradicts that C_{α_1} is a meet of P.

14. So $W \cap A \neq \emptyset$, for any $A \in P$.

15. Thus W is a meet of P, i.e. $W \in Q$.

16. Clearly,

$$W = \bigcap_{\alpha \in \bigwedge''} C_\alpha$$

is the least element of the chain $O = \{C_\alpha \mid \alpha \in \bigwedge''\}$.

17. By now, we have shown that every chain of Q has a least element.

18. By Zorn's Lemma, Q has a minimal element J. So J is a minimal meet of P.

Exercise 3.2 - Give a proof of lemma 3.14 based on definition 3.9 .

A. Trivial case: If ϕ is a formula, then

$$\forall x \forall y \phi \sim \forall y \forall x \phi - -(a)$$

$$\exists x \exists y \phi \sim \exists y \exists x \phi - -(b)$$

Proof (a)
As a preliminary note, consider the set expressed by
Var ($\forall x \forall y \phi$) = Var ($\forall y \forall x \phi$) = $X_1, X_2....X_m$ We can then use the following notation:

$$\forall x \forall y \phi = \forall x \forall y \phi(X_1, X_2, ...X_m; X, Y) - - - (1)$$
$$\forall y \forall x \phi = \forall y \forall x \phi(X_1, X_2, ...X_m; X, Y) - - - (2)$$

to make explicit that the variables $X_1, X_2, ...X_m$ have free occurences in ϕ Without loss of generality, we can assume that X and Y have no bound occurances in ϕ so that they are controlled by the two quantifiers shown in (1) and (2).
Let now A be an arbitary model of the language L , and let α be an arbitary A-assignment such that
$\alpha(x_i) = a_i$ with i = 1,2,....n and $a_i \epsilon A$
Suppose (left - to - right proof) that (1) is satisfied by α , i.e

$$\forall x \forall y \phi(a_1, a_2...a_m; X, Y) istrueinA - -(3)$$

Then if b and c are two arbitary elements of the universal instantiation steps on (3) We can assert that

$$\phi(a_1, a_2......a_m; X/b, Y/b) istrueinA - -4$$

Since b and c are completely arbitary and independent, we can first generalize on b and get

$$\forall x \phi(a_1, a_2, ...a_m; X, Y/c)$$

then generalize on c and obtain

$$\forall y \forall x \phi(a_1, a_2, ...a_m; X, Y)$$

Since the model A and the A-assignment α were arbitarily chosen, we can conclude that :

$$\models (\forall x \forall y \phi \rightarrow \forall y \forall x \phi)$$

The reasoning from right to left is exactly the same , so we finally have :

$$\models (\forall x \forall y \phi \leftrightarrow \forall y \forall x \phi)$$

i.e

$$\forall x \forall y \phi \sim \forall y \forall x \phi$$

The proof for (b) is similar , this time b and c are two particular elements of the universe, but again we can existsentially requantify on them in the first-in first-out order and reach the desired conclusion.

$$\exists x \exists y \phi \sim \exists y \exists x \phi$$

ii . If ϕ and ψ are formulas prove :

$$\models (\exists x \forall y \phi \rightarrow \forall y \exists x \phi)$$

(but $\forall y \exists x \rightarrow \exists x \forall y \phi$ is not valid)
Let α be an A-assignment such that $\exists x \forall y \phi$ is true. Then there exists an element b ϵ A such that
$\forall y \phi(a_1, a_2, ...a_m$; X/b , Y) is true in A furthermore, for any arbitary c ϵ A;
$\phi(a_1, a_2....a_m$; X/b, Y/c) is true in A.
We can now existsentially quantify on b and assert :
$\exists x \phi(a_1, a_2, ...a_m$; X, Y/c) is true in A since c is arbitary, we universally generalize on it and conclude:
$\forall y \exists x \phi(a_1, a_2......a_m$; X, Y) is true in A so, we see that any A-assignment that satisfies $\exists x \forall y \phi$ also satisfies $\forall y \exists x \phi$)
ie

$$\models (\exists x \forall y \phi \rightarrow \forall y \exists x \phi)$$

Let us now see what happens the other way aound, and assume that α is an arbitary assignment which satisfies $\forall y \exists x \phi$.

By universal instantiation, we first assert $\exists x\phi(a_1, a_2....a_m$; X, Y/b) is true in A, where b is an arbitary element of A. Now , the statement here above means that given an arbitary choice of b , we can find some c depending on that choice of b such that

$$\phi(a_1, a_2, ...a_m; X/c, Y/b)$$

is true in A.
Infomally, we can say that c = f(b).
In such situation, we cannot universally generalize on b , because it is not true that for a given c, the previous sentence holds for any choice of b. We therefore conclude that $\forall y\exists x\phi \rightarrow \exists x\forall y\phi$ is not valid.

iii Prove

$$\neg\forall x\phi \sim \exists x\neg\phi - -(a)$$

$$\neg\exists x\phi \sim \forall x\neg\phi - -(b)$$

a) Suppose again that α such that $\alpha(x_i) = a_i$ is an A-assignment satisfying $\neg\forall x\phi$;
then $\neg\forall x\phi(a_1, a_2...a_m$; X) is true in A.
then $\forall x\phi(a_1, a_2, ...a_m; X)$ is false in A. Then for some b ϵ A , $\phi(a_1, a_2, ...a_m$; X/b) is false and $\neg\phi(a_1, a_2...a_n$; X/b) is true. We can existsentially quantify this last sentence on the variable x and assert:
$\exists x\neg\phi(a_1, a_2, ...a_n$; X) is true in A since A, α were arbitarily chosen , we conclude:
$\models (\neg\forall x\phi \rightarrow \exists x\neg\phi\ )$

This validity of the implication from the right to the left is proven in the same fashion , leading to the desired conclusion $\neg\forall x\phi \sim \exists x\neg\phi$
The proof for (b) is similar.

iv Prove : If x is not free in ϕ_1 then $\forall x\phi \sim \exists x\phi \sim \phi$

If α is n arbitary A- assignment such that $\forall x\phi(or\exists x\phi)$ is satisfied and x is not free in ϕ_1, this amounts to say that the formula ϕ obtained from $\forall x\phi(or\exists x\phi)$ by dropping the left quantifier and replacing every occurance of x bound by such quantifier with any x ϵ A (or some a ϵ A) is satisfied by α. However since x is not free in ϕ_1 there is no such occurance and we see that trivally $\psi = \phi_1$. Therefore the formulas $\forall x\phi_1, \exists x\phi and\phi$ are all equivalent in this case.

v If ϕ_i (i ¡ m) are formulas, then

$$\forall x(\phi_0 \wedge .... \wedge \phi_{m-1})\ (\forall x\phi_0 \wedge ... \wedge \forall x\phi_{m-1}) - - - (a)$$

$$\exists x(\phi_0 \wedge .... \wedge \phi_{m-1})\ (\exists x\phi_0 \wedge ... \wedge \exists x\phi_{m-1}) - - - (b)$$

Proof (a)

N.B We use the notation $\phi\alpha$ given in the def 3.10 to indicate the sentence resulting from a formula ϕ via the A-assignment α .

Left - To - Right

Let α be an A-assignment such that [$\forall x(\phi_0 \wedge ..... \wedge \phi_{m-1}$] α is true in A. Then for any arbitary a ϵ A, we can assert :

$$\phi_0 x/a\alpha \wedge ... \wedge \phi_{m-1} x/a\alpha$$

is true in A

This amounts to say that evey conjunct of the sentence here above is true for an arbitary choice of an arbitary choice of a ϵ A i.e

for i ¡ n , ($\forall x\phi_i)\alpha$ is true in A

so we can assert :

$$[(\forall x\phi_0\alpha \wedge ... \wedge (\forall x\phi_{m-1})\alpha]$$

is true in A and finally

$$(\forall x\phi_0 \wedge ... \wedge \forall x\phi_{m-1})\alpha$$

is true in A

Hence, since A and α are arbitary

$$\models [\forall x(\phi_0 \wedge ... \wedge \phi_{m-1}) \rightarrow (\forall x\phi_0 \wedge ... \wedge \forall x\phi_{m-1})]$$

Right - to - Left

Let α be an A-assignment such that

$$(\forall x\phi_0 \wedge ... \wedge \forall x\phi_{m-1})\alpha$$

is true in A.

Then if a is an arbitary element of A, we can assert :

$$(\phi_0 x/a \wedge ... \wedge \phi_{m-1} x/a)\alpha$$

is true in A

We can then universalize a on the whole conjunct and conclude:

$$[\forall x(\phi_0 \wedge ...\phi_{m-1})]\alpha$$

is true in A.

and finally

$$\models [(\forall x\phi_0 \wedge ... \wedge \forall x\phi_{m-1} \rightarrow \forall x(\phi_0 \wedge ... \wedge \phi_{m-1})]$$

Hence part (a) is proven.

The proof for part (b) is similar.

1

vi If ϕ and ψ are formulas, prove

$$\models [\forall x(\phi \rightarrow \psi) \rightarrow (\forall x\phi \rightarrow \forall x\psi)]$$

— (a)

$$\models [\forall x(\phi \rightarrow \psi) \rightarrow (\exists x\phi \rightarrow \exists x\psi)]$$

— (b)

Proof (a)

Let α be an A-assignment such that $[\forall x(\phi \rightarrow \psi)]\alpha$ is true in A. Then, for any a ϵ A

$$[\phi x/a \rightarrow \psi x/a]\alpha is true in A ------(1)$$

$$\phi x/a\alpha \rightarrow \psi x/a\alpha is true in A ------(2)$$

Suppose now that $\forall x\phi$ is true in A ——- (3)

We then have in particular : $\phi x/a\alpha$ is true in A ——- (4)

And by (2) and (4)

$\phi x/a\alpha$ is true in A.

Since the initial choice of a in (1) was made arbitrarily, (4) holds for all a ϵ A. i.e $(\forall x\phi)\alpha$ is true in A.

So we can now assert :

$$[\forall x(\phi \rightarrow \psi)]\alpha \rightarrow [(\forall x\phi)\alpha \rightarrow (\forall x\phi)\alpha]$$

is true in A. i.e

$$\forall x(\phi \rightarrow \psi) \rightarrow (\forall x\phi \rightarrow \forall x\phi)]\alpha$$

is true in A. i.e

and from the arbitrarity of A and $alpha$:

$$\models [\forall x(\phi \rightarrow \psi) \rightarrow (\forall x\phi \rightarrow \forall x\phi)]$$

Proof (b) (Differs slightly from proof(a))

Let α be an A-assignment such that

$[\forall x(\phi \rightarrow \psi)]\alpha$ is true in A — (1)

and suppose that : $[\exists x\psi]\alpha$ is true in A — (2)

Then for some a ϵ A, $\phi X/a\alpha$ is true on A —(3)

If we select this particular element a ϵ A, we can get from (1)

$[\phi x/a \rightarrow \psi x/a]\alpha$ is true in A —— (4)

i.e $\phi x/a\alpha \rightarrow \psi x/a\alpha$ is true in A —— (5)

and by (3) and (5) $\phi x/a\alpha$ is true in A

From this, we can immediately conclude : $(\exists x\phi)\alpha$ is true in A.

So we now have :

$$[\forall x(\phi \to \psi)]\alpha \to [(\exists x\phi)\alpha \to (\exists x\psi)\alpha]$$

is true in A. i.e

$$[\forall x(\phi \to \psi) \to (\exists x\phi) \to (\exists x\psi)]\alpha$$

is true in A.
and from the arbitrariety of A and α :

$$\models [\forall x(\phi \to \psi) \to (\exists x\phi \to \exists x psi)]$$

The proof for part (vii) does not contain any intresting difference with respect to part (vi) above, so we do not report it here.

- Exercise 3.3 Prove : if ψ is a sentence and A a model, then we have A $\models \neg\phi$ iff A does not models ϕ. Give a simple example of a formula for which this is equivalence fails.
Proof :
Suppose that ϕ is a sentence and A $\models \neg\phi$, then $\neg\phi$ is a sentence, and as such it does not contain any free variables. Therefore, no matter what A-assignment α we choose $\neg\phi$ is true in A and ϕ is false in A and hence A does not models ϕ
Suppose now that A does not models ϕ , since ϕ is a sentence it can be either truein A or false in A, regardless of what A-assignment we consider. Hence in this case ϕ is false in A and $\neg\phi$ is true in A.
To say that $\neg\phi$ is true in it for all α amounts to say that A $\models \neg\phi$.
The following example shows that the property shown aabove does not hold for the formulas.
Let ϕ be the formulas P(x,a) and let's interpret ϕ in the following model A.
A = —N
P $\to >$
a $\to$ 1
Then we have the A-formula:
$\phi = x > 1$ and both
A $\not\models \neg(x > 1)$ and A $\not\models x > 1$.