The Largest Cartesian Closed Category of Stable Domains

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Berry's category of dI-domains with stable functions is a relatively intricate, yet elegant framework for semantics of programming languages. Despite over fifteen years of work in the area, the exact reasons for distributivity (Axiom d) and finitariness (Axiom I) have not been fully explicated. This paper shows that Axiom d and Axiom I are important when one works within the realm of Scott-domains. In particular, it has been shown that (i) if $[D \rightarrow_s D]$ has a countable basis, then D must be finitary, for any Scott-domain D; (ii) if $[D \rightarrow_s D]$ is bounded complete, then D must be distributive, for any finitary Scott-domain D. Therefore, the category of dI-domains is the largest cartesian closed category within omega-algebraic, bounded complete domains, with the exponential being the stable function space.

1 Introduction

Among Scott's many insights which shaped the whole area of domain theory, one is that the partial ordering of a domain should be interpreted as the ordering about information. "Thus," wrote Scott [16], "$x \sqsubseteq y$ means that x and y want to approximate the same entity, but y gives more information about it." If $x \sqsubseteq y$ means y contains more information than x, then an element containing a finite amount of information should only have a finite number of elements approximating it. The notion of a finite element is important because the most frequently used domains in semantics of programming languages are algebraic domains, each element of which is captured by approximating elements with finite information content.

One notices, however, that the standard pointwise order on continuous functions does not reflect the intuition about finite elements well. It is easy to find domains D and E which respect the intuition that finite elements represent finite information content, but there exist step functions (which are finite)

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dominating infinitely many step functions. Berry's stable order, on the other hand, seems to be the only way known to date which makes the function space respect the above intuition, although it was motivated from the study of sequentiality.

This paper studies Berry's category of dI-domains with stable functions, which is a relatively intricate, yet elegant framework for semantics of programming languages. This category was first shown in [2] to be cartesian closed and to provide a model for the typed λ-calculus. Since then, stable domains have found many applications, such as in linear logic [10,22], concurrency [19], polymorphism [4], and sequential computations [5].

A dI-domain is a bounded complete, ω-algebraic cpo which is distributive (Axiom d), and every finite element dominates a finite number of elements (Axiom I). Despite over fifteen years of work in the area, the exact reasons for distributivity and finitariness have not been fully explicated. This paper shows that Axiom d and Axiom I are important when one works within the realm of Scott-domains. The following is the main result of the paper.

Theorem 1 *Let D be a Scott-domain and $[D \to_s D]$ its stable function space. We have*

(i) if $[D \to_s D]$ has a countable basis, then D must be finitary, for any Scott-domain D;
(ii) if $[D \to_s D]$ is bounded complete, then D must be distributive, for any finitary Scott-domain D.

As an immediate consequence of this result, the category of dI-domains with stable maps is the largest cartesian closed category within omega-algebraic, bounded complete domains, with the exponential being the stable function space. These results reveal the deep reason why dI-domains must be what they are. It explains the phenomenon that a number of very recent papers on stable domains (*e.g.*, [1,7,9,14]) have already gone beyond Scott-domains. It implies that event domains [19,8] with stable functions are not cartesian closed, neither is the category of finitary quasi-prime algebraic domains [25] with stable mappings.

Previous work towards the understanding of stability includes [19], in which Winskel discussed a relationship between Axiom I and Scott's continuity thesis. In ([23], page 148), Zhang demonstrated that the stable function space $[\mathsf{I}_\infty \to_s \mathsf{I}_\infty]$ contains an uncountable basis (see Section 4 for a picture of I_∞). It was pointed out by Droste [7] and Amadio [1] that there are examples of non-distributive domains whose stable function space is not bounded complete. Taylor's unpublished manuscript [18] contains a categorical account of stability. It is clear that the result presented in this paper is the most general and complete.

Apart from the understanding of stability, it is perhaps worthwhile to put the results of this paper in a broader prospect. There are in general two kinds of results in domain theory. The first kind establishes certain properties, such as cartesian closure, existence of an universal object, etc., of a particular category of domains, demonstrating that these domains can serve as a suitable semantic space. Results on Scott-domains [11], SFP domains [15], and dI-domains [2] can be categorized as the first kind. The second kind of results has a different emphasis. Rather than establishing the suitability of a certain class of domains for semantic modeling, these results show that categories with certain properties just do not exist beyond a limit. The work of Smyth [17], Jung [13], and Huth [12] falls into the second kind. These results have been established by showing that a certain category is maximal, or the largest. Clearly, this paper belongs to the second list. It should be noted that, beyond Scott-domains, the category of L-domains with stable maps has been recently shown to be cartesian closed by Ehrhard and Malacaria [9] and Taylor [18]. Distributive L-domains with stable maps were investigated by Lamarche [14].

The rest of the paper is organized as follows. After a brief exposition of some preliminary stable domain theory in Section 2, we introduce the notion of trace sets in Section 3, which motivates the methodology used in the whole paper. Section 4 contains the proof that if $[D \to_s D]$ has countably many isolated elements, then D must be finitary (Theorem 10). Section 5 shows that distributivity is necessary for the stable function space to be bounded complete (Theorem 11). The paper ends with a couple of concluding remarks.

2 Stable Functions

In the rest of the paper, all domains are Scott-domains unless otherwise indicated. We briefly recall the basic definition of a stable function and the Berry order (the stable order), due to Berry [2].

A dI-domain is a consistently complete, ω-algebraic cpo D which satisfies

- Axiom d: $\forall x,\, y,\, z \in D.\, y \uparrow z \Longrightarrow x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$, and
- Axiom I: $\forall d \in D^0.\ \mid \{\, x \mid x \sqsubseteq d \,\} \mid < \infty$,

where D^0 is the set of isolated elements of D.

A Scott-domain is called finitary if it satisfies Axiom I. It is called distributive if it satisfies Axiom d. It is easy to see that D satisfies Axiom I if and only if $\{x \mid x \in D^0 \;\&\; x \sqsubset a\}$ is finite for every $a \in D^0$.

A function f from a Scott-domain D to a Scott-domain E is *stable* if it is continuous and preserves meets of pairs of compatible elements, *i.e.*,

$$\forall x,\ y \in D.\ \ x \uparrow y \Rightarrow f(x \sqcap y) = f(x) \sqcap f(y).$$

This is sometimes also called the stability condition. Let $f,\ g$ be in $[D \to_s E]$, the set of stable functions from D to E. f stably less than g, written as $f \sqsubseteq_s g$, if

$$\forall x,\ y \in D.\ \ x \sqsubseteq y \Rightarrow f(x) = f(y) \sqcap g(x).$$

Perhaps the most basic observation is that if D, E are Scott-domains, then $[D \to_s E]$ is a cpo. The following lemmas are well-known for the case of dI-domains, and proofs can be found in [23]. They are restated because here we use Scott-domains. However, the proofs given in [23] do not need distributivity or finitariness, so they work for the general case as well.

Lemma 2 *Let f and g be compatible stable functions in $[D \to_s E]$.*

(i) If $x,\ y$ are compatible in D, then

$$f(x) \sqcap g(y) = f(y) \sqcap g(x).$$

(ii) If $f(x) \sqsubseteq g(x)$ for every $x \in D$, then $f \sqsubseteq_s g$.

For a directed set of stable functions, the least upper bound exists and it is determined coordinate-wise.

Lemma 3 *If $F \subseteq [D \to_s E]$ is directed with respect to the stable order, then*

$$\bigsqcup F = \lambda x. \bigsqcup_{f \in F} f(x).$$

3 Trace Sets

Stable functions have a more subtle structure than the standard continuous functions. Step functions are not isolated in general, which makes them hard to work with. However, stable functions are determined by certain sets of pairs of isolated elements, to the effect that the stable order is captured by set inclusion on these sets. The idea of a trace originates from Berry [2]. Axiomatizations of traces for coherent spaces can be found in [10], and later generalized to dI-domains in [20,21]. Traces not only provide intuition on many results to follow, but they also motivates the key technique for the proofs.

Let $f : D \to E$ be a stable function. Define μf to be a set of pairs such that $(a,\ p) \in \mu f$ if

$$f(a) \sqsupseteq p \;\&\; [\forall a' \sqsubseteq a.\ f(a') \sqsupseteq p \Rightarrow a = a'],$$

where $a \in D^0$, the set of finitary elements of D and $p \in E^0$. We call μf the *trace* of f. Since μf always contains the special pair $(\perp_D, \perp_E)$, we simply remove it from traces for the rest of the paper.

For any $(a, p) \in \mu f$, we have $f(a) \sqsupseteq p$. Given any element x in D, if $x \sqsupseteq a$ and $(a, p) \in \mu f$, then $f(x) \sqsupseteq p$. Hence

$$f(x) \sqsupseteq \bigsqcup \{ \, p \mid \exists a \sqsubseteq x. \, (a, \, p) \in \mu f \, \}.$$

For the other direction of the inequality, however, note an important distinction from the case of dI-domains when traces are concerned. While for finitary Scott-domains it is true that for any q in E^0 such that $q \sqsubseteq f(x)$, there is a finite element $b \sqsubseteq x$ for which $(b, \, q) \in \mu f$, this no longer holds for general Scott-domains. The reason is that the minimal element b for which $q \sqsubseteq f(b)$ need not exist. As a consequence, not all stable functions can be captured by their traces. More precisely, different stable functions may have the same trace. This need not concern us, though, since we are interested in the construction of stable functions from *trace sets*, which are sets of pairs with the three properties mentioned in (ii) of the following theorem.

Theorem 4 *Let $f, \, g : D \to E$ be stable functions, where D and E are Scott-domains. We have*

(i) If f and g are compatible, with $(a, p) \in \mu f$ and $p \sqsubseteq g(a)$, then $(a, p) \in \mu g$.
(ii) A set $\{ \, (a_i, \, p_i) \mid i \in I \, \} \subseteq D^0 \times E^0$ is the trace of a stable function if it has the following three properties (where $\subseteq^{fin}$ stands for finite subset):
- *Compatibility: $\forall J \subseteq^{fin} I. \, \{ \, a_i \mid i \in J \, \}\!\uparrow \Rightarrow \{ \, p_i \mid i \in J \, \}\!\uparrow$,*
- *Minimality: $a_i \uparrow a_j \;\&\; (p_i = p_j) \;\Rightarrow\; (a_i = a_j)$, and*
- *Completeness: For every finite subset J of I,*

$$\begin{aligned} &\{a_j \mid j \in J\}\!\uparrow \;\; \& \; p \sqsubseteq \bigsqcup \{p_j \mid j \in J\} \\ &\Rightarrow \exists a \sqsubseteq \bigsqcup \{a_j \mid j \in J\}. \, (a, p) \in \{(a_i, p_i) \mid i \in I\}. \end{aligned}$$

Note that this result is similar in spirit to Theorem 3.1 in [21]. The major difference is that not all traces of stable functions are trace sets, for lack of the completeness property.

Completeness, nevertheless, is helpful. For any trace set $\{ \, (a_i, \, p_i) \mid i \in I \, \}$, the set

$$\{p_i \mid a_i \sqsubseteq x \;\&\; i \in I\}$$

is directed, given any x in D. (It is clearly bounded, because of compatibility.) Indeed, for any $p_m, \, p_n$ in this set, we have

$$p_m \sqcup p_n \sqsubseteq \bigsqcup \{p_i \mid a_i \sqsubseteq x \;\&\; i \in I\}.$$

Since $p_m \sqcup p_n$ is a compact element,

$$p_m \sqcup p_n \sqsubseteq \bigsqcup\{p_j \mid j \in J\}$$

for some finite subset J of I such that $a_j \sqsubseteq x$ for any $j \in J$. By completeness, there is some a with

$$a \sqsubseteq \bigsqcup\{a_j \mid j \in J\}$$

and

$$(a, p_m \sqcup p_n) \in \{\ (\,a_i,\ p_i\,) \mid i \in I\ \}.$$

Hence,

$$p_m \sqcup p_n \in \{p_i \mid a_i \sqsubseteq x \ \&\ i \in I\}.$$

Clearly, if (a, b) is a member of a trace set T, then $\mathsf{T} \cap (a{\downarrow} \times b{\downarrow})$ determines a monotonic function from $b{\downarrow}$ to $a{\downarrow}$ just by reversing the pairs in the set, where

$$y{\downarrow} := \{x \mid x \sqsubseteq y\}.$$

Proof of (i) Suppose f and g are bounded above, and (a, p) is a pair in μf with $p \sqsubseteq g(a)$. For any $b \sqsubseteq a$, if $p \sqsubseteq g(b)$, then we have

$$p \sqsubseteq f(a) \sqcap g(b) = f(b) \sqcap g(a) \sqsubseteq f(b),$$

by Lemma 2. But a is minimal for f to attain p, so $b = a$. This implies a is minimal for g to attain p, so (a, p) is a member of μg, too.

Proof of (ii) Let $\{\,(a_i,\ p_i) \mid i \in I\,\} \subseteq D^0 \times E^0$ be a trace set. We show that the stable function f for which

$$\{\ (\,a_i,\ p_i\,) \mid i \in I\ \} = \mu f$$

can be obtained as the *pointwise* lub $\bigsqcup_{i \in I} [\,a_i, p_i\,]$, where

$$[\,a, p\,](x) = \begin{cases} p & \text{if } x \sqsupseteq a\ , \\ \bot & \text{otherwise.} \end{cases}$$

It is clear that $\bigsqcup_{i \in I} [\,a_i, p_i\,]$ is continuous.

To check stability let $x, y \in D$ and $x \uparrow y$. Suppose

$$p \sqsubseteq \bigsqcup_{i \in I} [\,a_i, p_i\,](x)\ \sqcap\ \bigsqcup_{i \in I} [\,a_i, p_i\,](y)$$

where p is an isolated element in E. We have

$$p \sqsubseteq \bigsqcup\{p_i \mid i \in I \mathbin{\&} a_i \sqsubseteq x\}.$$

Since p is isolated, there is a finite set J of I, such that $p \sqsubseteq \bigsqcup\{p_j \mid j \in J\}$. By completeness, this implies the existence of some $a \sqsubseteq x$ such that

$$(a, p) \in \{(a_i, p_i) \mid i \in I\}.$$

Similarly, there exists some $b \sqsubseteq y$ such that

$$(b, p) \in \{(a_i, p_i) \mid i \in I\}.$$

By minimality, we have $a = b$, and so $a \sqsubseteq x \sqcap y$. Therefore,

$$p \sqsubseteq \bigsqcup_{i \in I} [\, a_i, p_i \,](x \sqcap y).$$

Since E is algebraic, we have

$$\bigsqcup_{i \in I} [\, a_i, p_i \,](x \sqcap y) \sqsupseteq \bigsqcup_{i \in I} [\, a_i, p_i \,](x) \ \sqcap \ \bigsqcup_{i \in I} [\, a_i, p_i \,](y).$$

This implies that $\bigsqcup_{i \in I} [\, a_i, p_i \,]$ is stable.

It remains to show that

$$\{\ (a_i,\ p_i) \mid i \in I\ \} = \mu f,$$

where we abbreviate $\bigsqcup_{i \in I} [\, a_i, p_i \,]$ as f.

– ($\subseteq$) For any (a_j, p_j) $(j \in I)$, we have

$$\begin{aligned} f(a_j) &= \bigsqcup\{\ p_i \mid a_i \sqsubseteq a_j\ \} \\ &\sqsupseteq p_j. \end{aligned}$$

Let y be an element in E such that $y \sqsubseteq a_j$ and $f(y) \sqsupseteq p_j$. As

$$f(y) = \bigsqcup\{\ p_i \mid a_i \sqsubseteq y \mathbin{\&} i \in I\ \}$$

and p_j is isolated,

$$p_j \sqsubseteq \bigsqcup\{p_i \mid i \in J\}$$

for some finite set J such that $a_i \sqsubseteq y$ for each i in J. By completeness, there is some (a_k, p_k) $(k \in I)$ such that $p_k = p_j$ and $a_k \sqsubseteq \bigsqcup\{a_i \mid i \in J\}$. This means $a_k \sqsubseteq y$. By minimality, then, we have $a_k = a_j$, and so $y = a_j$. Thus for any $j \in I$, (a_j, p_j) is a member in μf.

– ($\supseteq$) For any $(a,\ p) \in \mu f$, we have $f(a) \sqsupseteq p$. We have

$$\begin{aligned}(a,p) \in \mu f &\Rightarrow p \sqsubseteq \bigsqcup\{p_i \mid a_i \sqsubseteq a \ \&\ i \in I\} \\ &\Rightarrow \exists J \subseteq^{fin} I\ p \sqsubseteq \bigsqcup\{p_j \mid j \in J\} \ \&\ [\forall j \in J\ a_j \sqsubseteq a] \\ &\Rightarrow \exists k \in I\ [a_k \sqsubseteq a \ \&\ p_k = p] \qquad (\text{ by completeness}) \\ &\Rightarrow p \sqsubseteq f(a_k) \\ &\Rightarrow a_k = a \qquad (\text{ for } \ (a,\ p) \in \mu f) \\ &\Rightarrow (a,p) \in \{(a_i, p_i) \mid i \in I\}.\end{aligned}$$

□

4 Finiteness

In this section we introduce a sequence of lemmas leading to a proof that ω-algebraicity is preserved for the stable function space only if finiteness of the domains are satisfied. We briefly mention a cover property about isolated elements first, however.

An element y is *covered by* x if $y \sqsubset x$ and $y \sqsubseteq z \sqsubseteq x$ implies $y = z$ or $z = x$. Write $\lfloor x \rfloor$ for the set of elements covered by x. Call $\lfloor x \rfloor$ the *lower cover* of x. A lower cover $\lfloor x \rfloor$ is *complete* if every element strictly below x is below some element in $\lfloor x \rfloor$. Note that if $\lfloor x \rfloor$ is complete, then either $x = \bot$, or $\lfloor x \rfloor \neq \emptyset$.

Lemma 5 *Let d be an isolated element of a dcpo D. Then $\lfloor d \rfloor$ is complete.*

PROOF. We use Zorn's lemma. Let $x \sqsubset d$. Consider the subset $[x, d)$, where

$$[x, d) := \{y \mid x \sqsubseteq y \sqsubset d\}.$$

With the order of D restricted to this set, we get a partial order. Let L be any chain form $[x, d)$. Because L is directed, $\bigsqcup L$ exists in D. Clearly $\bigsqcup L \sqsubseteq d$. The equality cannot occur since d is isolated. Therefore L has an upper bound in $[x, d)$. By Zorn's lemma, $[x, d)$ has a maximal element. It is easy to see that this maximal element belongs to $\lfloor d \rfloor$, and it is above x. Therefore, $\lfloor d \rfloor$ is complete.

□

We now import the concept of sublattices from lattice theory [6], with a slight modification to reflect algebraicity, to the theory of domains. Let D be a Scott-

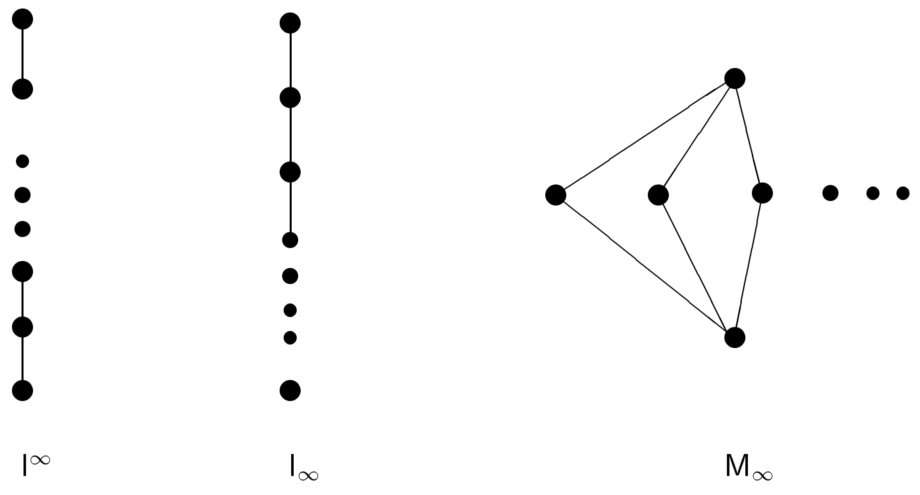

Fig. 1. Typical domains violating Axiom I

domain and let M be a non-empty subset of D. Then M is a *subdomain* of D if M is a Scott-domain inheriting the ordering of D in such as way that

- an element in M is isolated with respect to M if and only it is isolated in D, and
- for any non-empty subset X of M, if $\bigsqcap X$ exists in D, then $\bigsqcap X \in M$, and if $\bigsqcup X$ exists in D, then $\bigsqcup X \in M$.

We write $M \hookrightarrow D$ to indicate that D has a subdomain isomorphic to M.

Picture 1 contains three typical Scott-domains violating Axiom I.

Lemma 6 *If D is a Scott-domain which does not satisfy Axiom I, then*

$$\textit{either } \mathsf{I}^\infty \hookrightarrow D, \quad \textit{or } \mathsf{I}_\infty \hookrightarrow D, \quad \textit{or } \mathsf{M}_\infty \hookrightarrow D.$$

PROOF. For any domain D, we write $a\!\downarrow$ for the set $\{x \mid x \in D \;\&\; x \sqsubseteq a\}$, and $a\!\uparrow$ for the set $\{x \mid x \in D \;\&\; x \sqsupseteq a\}$.

Suppose a is an isolated element of D such that $\{x \mid x \in D^0 \;\&\; x \sqsubset a\}$ is an infinite set. By Lemma 5, $\lfloor a \rfloor$ is complete. If any element in $a\!\downarrow$ is not isolated, then $\mathsf{I}^\infty \hookrightarrow D$. So let's assume that $a\!\downarrow \subseteq D^0$. If an element a_1 in $\lfloor a \rfloor$ dominates an infinite number of isolated elements, we consider $\lfloor a_1 \rfloor$, and so on. If it is possible to continue indefinitely like this, then there is an infinite sequence of isolated elements $a_1, a_2, a_3, \cdots$ such that $a_{i+1} \in \lfloor a_i \rfloor$ for each $i \geq 1$. Moreover, the greatest lower bound of this chain is an isolated element. In this case $\mathsf{I}_\infty \hookrightarrow D$.

If neither $\mathsf{I}^\infty \hookrightarrow D$ nor $\mathsf{I}_\infty \hookrightarrow D$, we must have, at some point in the process, reached an isolated element h_0 for which the lower cover $\lfloor h_0 \rfloor$ is an infinite set, every element in it is isolated, and every element in it dominates a finite number of elements. We must have $\mathsf{M}_\infty \hookrightarrow D$ for the following reasons. Consider the principal ideal $h_0 \downarrow$. We can regard the principal ideal as an infinite tree with the bottom as the root, and with "shared subtrees teared apart". As $h_0 \downarrow$ does not contain an infinite branch, we have, by König's lemma, an element $h_1 \in h_0 \downarrow$ such that there is an infinite antichain $\{x_i \mid i \geq 1\}$ in $h_0 \downarrow$, each member of which covers h_1. Moreover, we can assume that the upward closure of each x_i in $h_0 \downarrow$ is finite. Let h_2 be x_1. Let h_3 be an x_i whose index is bigger than that of any element in

$$\{x_i \mid i \geq 1\} \cap (h_2 \uparrow) \downarrow,$$

where the closures are taken with respect to $h_0 \downarrow$. This is possible because

$$\{x_i \mid i \geq 1\} \cap (h_2 \uparrow) \downarrow$$

is a finite set. Similarly, pick up the next, h_4 to be an x_i whose index is bigger than that of any element in

$$\{x_i \mid i \geq 1\} \cap (h_3 \uparrow) \downarrow,$$

and so on. (One can picture this process by considering the finite area covered by the radio wave from some source after reflected from the atmosphere.) This gives us the infinite set $\{h_0, h_1, h_2, h_3, \cdots\}$, which is a subdomain of D isomorphic to M_∞.

□

Lemma 7 *Let D be a Scott-domain and $a_0 \sqsupset a_1 \sqsupset a_2 \cdots$ be an infinite chain of isolated elements of D. If $t : \omega \to \omega$ is a strict, one-one, and monotonic function with respect to the nature order of integers, then*

$$F_t := \bigsqcup_{i \geq 0} [a_{t(i)}, a_i]$$

is an isolated stable function in $[D \to_s D]$. Moreover, for any stable function $f : D \to D$ such that

- *f is compatible with F_t, and*
- *$f(a_{t(i)}) \sqsupseteq a_i$ for each $i \geq 0$,*

we have $f \sqsupseteq_s F_t$.

PROOF. Since the set $\{(a_{t(i)}, a_i) \mid i \geq 0\}$ satisfies compatibility (Theorem 4), F_t is a continuous function. To check the stability of F_t, let x, y be compatible elements in D. For any $d \in D^0$ below $F_t(x) \sqcap F_t(y)$, we have

$$d \sqsubseteq \bigsqcup \{a_k \mid a_{t(k)} \sqsubseteq x\}$$

and

$$d \sqsubseteq \bigsqcup \{a_k \mid a_{t(k)} \sqsubseteq y\}.$$

If d is not the bottom, then $d \sqsubseteq a_k$ and $d \sqsubseteq a_l$ for some $a_{t(k)}$ below x and $a_{t(l)}$ below y. Since $\{a_i \mid i \in \omega\}$ is a chain, one of $a_{t(k)}$ or $a_{t(l)}$ must be below both x and y. Therefore, $d \sqsubseteq F_t(x \sqcap y)$, sufficient for stability to hold.

For the second part, we use (i) of Theorem 4 and show

$$\{(a_{t(i)}, a_i) \mid i \geq 0\} \subseteq \mu F_t.$$

Note that

$$\begin{aligned} F_t(a_{t(j)}) &= \sqcup \{ a_i \mid a_{t(i)} \sqsubseteq a_{t(j)} \} \\ &\sqsupseteq a_j. \end{aligned}$$

Let y be an element in D such that $y \sqsubseteq a_{t(j)}$ and $F_t(y) \sqsupseteq a_j$. Without loss of generality, we may assume that $a_j \neq \bot$. Hence, there exists some a_i such that $a_j \sqsubseteq a_i$ and $a_{t(i)} \sqsubseteq y \sqsubseteq a_{t(j)}$. By the monotonicity of t, this is only possible when $y = a_{t(j)}$. Thus for any $j \geq 0$, $(a_{t(j)}, a_j)$ is a member in μF_t.

That any stable f compatible with F_t and whose trace set contains

$$\{(a_{t(i)}, a_i) \mid i \geq 0\}$$

must dominate F_t follows from Theorem 4 and the fact that F_t is the least in the extensional order among those continuous functions F such that

$$F(a_{t(i)}) \sqsupseteq a_i$$

for every $i \geq 0$. Here we have implicitly used Lemma 2.

In the rest of the proof, we show that all the stable functions F_t are isolated. Let $\{\phi_i \mid i \in I\}$ be a directed set of stable functions such that

$$\bigsqcup_{i \in I} \phi_i \sqsupseteq_s F_t.$$

By Lemma 3, we have

$$\bigsqcup_{i \in I} \phi_i(a_0) \sqsupseteq F_t(a_0).$$

Therefore, there exists $i_0 \in I$ such that $\phi_{i_0}(a_0) \sqsupseteq F_t(a_0)$, as $F_t(a_0) = a_0$ is isolated. Since ϕ_{i_0} and F_t are compatible, we have, by Lemma 2, for all $k \geq 1$, $F_t(a_k) \sqcap \phi_{i_0}(a_0) = \phi_{i_0}(a_k) \sqcap F_t(a_0)$. Hence, for all $k \geq 1$,

$$\begin{aligned} F_t(a_k) &= F_t(a_k) \sqcap F_t(a_0) \\ &= F_t(a_k) \sqcap \phi_{i_0}(a_0) \\ &= \phi_{i_0}(a_k) \sqcap F_t(a_0) \\ &\sqsubseteq \phi_{i_0}(a_k). \end{aligned}$$

By the first part of the proof for Lemma 7, we have $F_t \sqsubseteq_s \phi_{i_0}$, which shows that F_t is isolated in the stable function space.

□

The proof for the next lemma is extremely similar to the previous one, hence omitted.

Lemma 8 *Let D be a Scott-domain and $a_1 \sqsubset a_2 \sqsubset \cdots$ be an infinite chain of isolated elements of D such that $a_i \sqsubset a_0$ for $i \geq 1$, where a_0 is isolated. If $t : \omega \to \omega$ is a strict, one-one, and monotonic function, then*

$$F_t := \bigsqcup_{i \geq 0} [a_{t(i)}, a_i]$$

is an isolated stable function in $[D \to_s D]$. Moreover, for any stable function $f : D \to D$ such that

- *f is compatible with F_t, and*
- *$f(a_{t(i)}) \sqsupseteq a_i$ for each $i \geq 0$,*

we have $f \sqsupseteq_s F_t$.

The next lemma is of a slightly different flavor with a slightly different proof. It treats the case of an infinite antichain of isolated elements.

Lemma 9 *Let $b_0, b_1 \in D^0$ and let $\{b_2, b_3, b_4, \cdots\}$ be an infinite antichain of isolated elements such that every b_i ($i \geq 2$) covers b_1, and $b_0 = b_i \sqcup b_j$ if $i \neq j$ and $i, j \geq 2$. Then for each one-one function $t : \omega \to \omega$ with $t(0) = 0$ and $t(1) = 1$,*

$$F_t := \bigsqcup_{i \geq 0} [b_{t(i)}, b_i]$$

is an isolated stable function in $[D \to_s D]$. Moreover, for any stable function $f : D \to D$ such that

- *f is compatible with F_t, and*

- $f(b_{t(i)}) \sqsupseteq b_i$ *for each* $i \geq 0$,

we have $f \sqsupseteq_s F_t$.

PROOF. We check the stability of F_t first. Let x, y be compatible elements in D. For any non-bottom, isolated element d below $F_t(x) \sqcap F_t(y)$, we have

$$d \sqsubseteq \bigsqcup\{b_k \mid b_{t(k)} \sqsubseteq x\}$$

and

$$d \sqsubseteq \bigsqcup\{b_k \mid b_{t(k)} \sqsubseteq y\}.$$

Both $\{b_k \mid b_{t(k)} \sqsubseteq x\}$ and $\{b_k \mid b_{t(k)} \sqsubseteq y\}$ must be non-empty. There are four possibilities:

(i) At least one of $\{b_k \mid b_{t(k)} \sqsubseteq x\}$ or $\{b_k \mid b_{t(k)} \sqsubseteq y\}$ contains exactly one element.

(ii) Both $\{b_k \mid b_{t(k)} \sqsubseteq x\}$ and $\{b_k \mid b_{t(k)} \sqsubseteq y\}$ contain exactly two elements, but the elements they contain are the same.

(iii) Both $\{b_k \mid b_{t(k)} \sqsubseteq x\}$ and $\{b_k \mid b_{t(k)} \sqsubseteq y\}$ contain exactly two elements, but the elements they contain are not the same.

(iv) At least one of $\{b_k \mid b_{t(k)} \sqsubseteq x\}$ or $\{b_k \mid b_{t(k)} \sqsubseteq y\}$ contains infinitely many elements.

Cases (i), (ii), (iv) readily imply $d \sqsubseteq F_t(x \sqcap y)$. For Case (iii), we must have $x \sqsupseteq b_{t(i)}$, $y \sqsupseteq b_{t(j)}$, and $x \sqcap y = b_1$ for some $i \neq j$ such that $i, j \geq 2$. This implies

$$d \sqsubseteq b_i \sqcap b_j = b_1.$$

Clearly, $d \sqsubseteq F_t(x \sqcap y)$. This proves that F_t is stable.

In light of (i) of Theorem 4, $f \sqsupseteq_s F_t$ if

$$\{(b_{t(i)}, b_i) \mid i \geq 0\} \subseteq \mu F_t,$$

so we prove it now. Note that

$$\begin{aligned} F_t(b_{t(j)}) &= \bigsqcup\{\, b_i \mid b_{t(i)} \sqsubseteq b_{t(j)} \,\} \\ &\sqsupseteq b_j. \end{aligned}$$

Let y be an element in D such that $y \sqsubseteq b_{t(j)}$ and $F_t(y) \sqsupseteq b_j$. The case when $j = 0$ or 1 is easy and we assume $j \geq 2$. Because $b_j \neq \bot$, there exists some b_i such that $b_j \sqsubseteq b_i$ and $b_{t(i)} \sqsubseteq y \sqsubseteq b_{t(j)}$. However, b_i and b_j are incomparable if $i \neq j$. So, $i = j$ and $y = b_{t(j)}$. Thus for any $j \geq 0$, $(b_{t(j)}, b_j)$ is a member in μF_t.

That any stable f compatible with F_t and whose trace set contains

$$\{(b_{t(i)}, b_i) \mid i \geq 0\}$$

must dominate F_t again follows from the fact that F_t is the least continuous function in the usual extensional order among those continuous functions F such that $F(b_{t(i)}) \sqsupseteq b_i$ for every $i \geq 0$.

We now prove that all the stable functions F_t are isolated. Let $\{\phi_i \mid i \in I\}$ be a directed set of stable functions such that

$$\bigsqcup_{i \in I} \phi_i \sqsupseteq_s F_t.$$

By Lemma 3, we have

$$\bigsqcup_{i \in I} \phi_i(b_0) \sqsupseteq F_t(b_0).$$

Therefore, there exists $i_0 \in I$ such that $\phi_{i_0}(b_0) \sqsupseteq F_t(b_0)$, as $F_t(b_0) = b_0$ is isolated. Since ϕ_{i_0} and F_t are compatible, we have, by Lemma 2, for all $k \geq 1$, $F_t(b_k) \sqcap \phi_{i_0}(b_0) = \phi_{i_0}(b_k) \sqcap F_t(b_0)$. Hence, for all $k \geq 1$,

$$\begin{aligned} F_t(b_k) &= F_t(b_k) \sqcap F_t(b_0) \\ &= F_t(b_k) \sqcap \phi_{i_0}(b_0) \\ &= \phi_{i_0}(b_k) \sqcap F_t(b_0) \\ &\sqsubseteq \phi_{i_0}(b_k). \end{aligned}$$

This implies $F_t \sqsubseteq_s \phi_{i_0}$, which shows that F_t is isolated in the stable function space.

□

Theorem 10 *Let D be a Scott-domain such that $[D \to_s D]$ has a countable basis. Then D must satisfy Axiom I.*

PROOF. Suppose D does not satisfy Axiom I. We show that there are uncountably many isolated elements in the stable function space $[D \to_s D]$.

By Lemma 6, one of I^∞, I_∞, or M_∞ must be a sub-domain of D. Lemma 7, Lemma 8, and Lemma 9 imply that no matter which case it is, one can construct, for each strict, one-one, and monotonic function $t : \omega \to \omega$, a distinct, isolated stable function F_t. (Note that strictness does not affect countability here.) It suffices to show that there are uncountably many such t's, and we use the *diagonal argument*. To arrive at a contradiction, we assume that there are countably many such t's and label all such functions as $t_1, t_2, t_3, \cdots$. Furthermore, we would be able to list the elements of the image of each t_i as

$$(t_i(1), t_i(2), t_i(3), \cdots, t_i(k), \cdots),$$

with the infinite list increasing. Now consider the infinite list

$$(r(1), r(2), r(3), \cdots, r(k), \cdots),$$

with $r(i)$ $(i \geq 1)$ inductively defined as:

$$\begin{aligned} r(1) &:= t_1(1) + 1, \\ r(i+1) &:= t_{i+1}(i+1) + r(i). \end{aligned}$$

This list is clearly strictly increasing. The function r defined by this list is distinct from every t_i, $i \geq 1$, because for each i, $t_i(i) < r(i)$. Therefore, the list t_i can not exhaust all increasing functions; a contradiction.

□

5 Distributivity

The objective of this section is to show that the distributivity condition is necessary for preserving bounded completeness of the stable function space. This is stated more precisely in the following theorem. In light of the results of the previous section, we restrict ourselves to finitary Scott-domains in this section.

Theorem 11 *Let D be a finitary Scott-domain such that $[D \to_s D]$ is consistently complete. Then D must be distributive.*

It is useful to recall the well-known observation (cf. [23] page 140) that Axiom d can be replaced by a seemingly weaker one, requiring distributivity to hold only for compatible triples: D satisfies Axiom d if and only if for all $x, y, z \in D$

$$\{x, y, z\}\uparrow \implies x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z).$$

This observation tells us that if the distributivity law fails for a Scott-domain, then it fails in one of its principal ideals $d\downarrow$.

Moreover, if the distributivity law fails, then it fails for some isolated elements x, y, z. This is because for increasing chains

$$a_0 \sqsubseteq a_1 \sqsubseteq a_2 \sqsubseteq \cdots,$$

$$b_0 \sqsubseteq b_1 \sqsubseteq b_2 \sqsubseteq \cdots,$$

and

$$c_0 \sqsubseteq c_1 \sqsubseteq c_2 \sqsubseteq \cdots$$

in a Scott-domain, we have

$$(\bigsqcup_{i\in\omega} a_i) \sqcap (\bigsqcup_{i\in\omega} b_i) = \bigsqcup_{i\in\omega} a_i \sqcap b_i$$

and

$$(\bigsqcup_{i\in\omega} a_i) \sqcap ([\bigsqcup_{i\in\omega} b_i] \sqcup [\bigsqcup_{i\in\omega} c_i]) = \bigsqcup_{i\in\omega} (a_i \sqcap [b_i \sqcup c_i]).$$

Therefore, if

$$(x \sqcap y) \sqcup (x \sqcap z) \sqsubset x \sqcap (y \sqcup z),$$

then we can replace each x, y, z by least upper bounds of increasing chains of isolated elements to obtain some isolated a_i, b_i, c_i such that

$$(a_i \sqcap b_i) \sqcup (a_i \sqcap c_i) \sqsubset a_i \sqcap (b_i \sqcup c_i).$$

We have, in effect, shown the following.

Lemma 12 *If the distributivity law fails in a finitary Scott-domain D, then it fails in a finite principal ideal $p\!\downarrow$, where $p \in D^0$.*

We now invoke the classical result in Lattice Theory (see, *e.g.*, [3,6]), sometimes called the M_3-N_5 Theorem. It says that a lattice L is non-distributive if and only if $\mathsf{M}_3 \hookrightarrow L$ or $\mathsf{N}_5 \hookrightarrow L$ (see Figure 2).

This proves the following lemma.

Lemma 13 *If D is a non-distributive, finitary Scott-domain, then*

$$\textit{either } \mathsf{M}_3 \hookrightarrow D, \quad \textit{or } \mathsf{N}_5 \hookrightarrow D.$$

We are now in a position to prove Theorem 11.

Proof of Theorem 11 Suppose D is a non-distributive, finitary Scott-domain. By the previous lemma, either $\mathsf{M}_3 \hookrightarrow D$, or $\mathsf{N}_5 \hookrightarrow D$. In any of the two cases, consider the trace sets $\{(x, x) \mid x \sqsubseteq u\}$ and $\{(x, x) \mid x \sqsubseteq v\}$. By Theorem 4, they determine stable functions in $[D \rightarrow_s D]$. There are at least two possible,

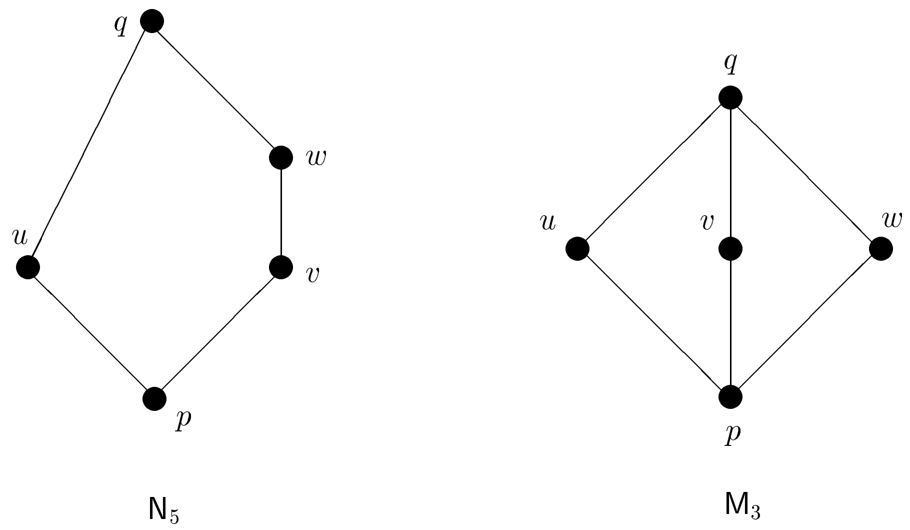

Fig. 2. Two basic non-distributive lattices

minimal ways to make the union of the trace sets complete. One is to add pairs (q, q) and (w, w), and the other is to add pairs (q, q) and (q, w). This gives rise to incompatible minimal upper bounds. Therefore, $[D \to_s D]$ is *not bounded complete.*

□

For those who would like to think in terms of step functions, the stable functions (here, $\sqcup$ is the pointwise lub) $[u, u] \sqcup [p, p]$ and $[v, v] \sqcup [p, p]$ in $[D \to_s D]$ have two minimal upper bounds, which are

$$[u, u] \sqcup [v, v] \sqcup [p, p]$$

and

$$[u, u] \sqcup [v, v] \sqcup [p, p] \sqcup [w, w].$$

6 Conclusion

By showing that dI-domains and stable maps form the largest cartesian closed category within Scott-domains, we have achieved a complete understanding of Axiom d and Axiom I. The crucial step in the proof of this result is to first establish the fact that the ω-algebraicity of $[D \to_s D]$ implies the finitariness of D, by employing the trace technique and an important observation from [23]. Once this established, the proof of distributivity of D from the bounded-completeness of $[D \to_s D]$ becomes relatively simple, for we may assume D to be finitary. We believe that if the finiteness condition is dropped, Theorem 11 still holds, although the proof becomes complicated because Lemma 13 can no longer be used.

The category of dI-domains with linear, stable maps has recently been shown to be monoidal closed [22]. It is expected that similar techniques might be used to show that this is the largest monoidal closed category within Scott-domains, in the sense explained in [12].

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