Disjunctive Systems and L-Domains

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Abstract. Disjunctive systems are a representation of L-domains. They use sequents of the form $X \vdash Y$, with X finite and Y pairwise disjoint. We show that for any disjunctive system, its elements ordered by inclusion form an L-domain. On the other hand, via the notion of stable neighborhoods, every L-domain can be represented as a disjunctive system. More generally, we have a categorical equivalence between the category of disjunctive systems and the category of L- domains. A natural classification of domains is obtained in terms of the style of the entailment: when $|X| = 2$ and $|Y| = 0$ disjunctive systems determine coherent spaces; when $|Y| \leq 1$ they represent Scott domains; when either $|X| = 1$ or $|Y| = 0$ the associated cpos are distributive Scott domains; and finally, without any restriction, disjunctive systems give rise to L-domains.

1 Introduction

Discovered by Coquand [Co90] and Jung [Ju90] independently, L-domains form one of the maximal cartesian closed categories of algebraic cpos. Together with the work of Smyth [Sm83] which shows that the SFP domains of Plotkin [Pl76] is the largest cartesian closed category inside the ω-algebraic cpos, we have a better picture of good categories of domains that may be used in denotational semantics of programming languages.

The primary contribution of this paper is to give a representation of L-domains, based concretely on sets and relations. These concrete structures, which are called *disjunctive systems*, provide a general framework encompassing coherent spaces, Scott domains, and bifinite L-domains. Our representation follows the idea of Scott [Sc82] in his work on information systems, and hence it inherits the benefits of these structures.

Previous work has shown that Scott domains, SFP domains, and stable domains can all be represented by structures similar to information systems ([LaWi84], [Sc82], [Zh90b], [Zh89]). The topic of this paper comes very naturally along this line of research. It adds to our belief that *good categories of domains all have nice concrete representations.*

A disjunctive system is a structure with sequents of the form $X \vdash Y$ on propositions (tokens) about computation. These sequents generalize the usual notion of sequents due to Gentzen: Although X is required to be finite (and non-empty), Y need not be so. A sequent $X \vdash Y$ should read 'conjunction of X's entails disjunction of Y's'. The name 'disjunctive' originates from the work of Johnstone [Jo77]. It reflects the condition that whenever we write $X \vdash Y$, Y must be pairwise disjoint, in the sense that two distinct propositions in Y always contradict each other. In terms of sequents, this is expressed by

$$\forall a,\, b \in Y.\; a \neq b \Longrightarrow \{a,\, b\} \vdash \emptyset.$$

Perhaps *exclusive-or* is the most familiar example of 'disjunctiveness'. However, situations like this arise frequently in computation. For example, although a storage can hold different data, it can only store one datum at a time. Essentially, anything which has to do with resources involves the notion of disjunctiveness. Suppose a candy bar, a pack of cigarette, and a cup of coffee cost one dollar each. With one dollar, you can buy a candy bar, a pack of cigarette, or a cup of coffee. However, you can only buy one of them.

The notion of disjunctiveness has appeared already in different areas of theoretical study. It was considered in [Die76] and [Jo77] in the context of category theory. When looking for a topological characterization of stable functions of Berry [Be78], one naturally arrives at stable neighborhoods [Zh], [Zh90a], those open sets which have the disjoint property (see Section 2).

Our purpose in considering disjunctive systems is to study concrete representation of L-domains and its consequences. The concept of *elements* serves as a bridge between disjunctive systems and domains. An element of a disjunctive system is a set x of propositions such that whenever $X \vdash Y$ for some $X \subseteq x$, $Y \cap x$ is non-empty. Ordered by set inclusion these elements form a cpo, with $\emptyset$ the bottom. We are going to show that disjunctive systems represent L-domains, those algebraic cpos in which every principal ideal $\downarrow x$ is a complete lattice. This main result is expressed as the following two theorems, which is generalized to the equivalence of the related categories in Section 6.

Theorem 1.1 *For a disjunctive system $\underline{A}$, the set of elements $|\underline{A}|$ ordered by inclusion is an L-domain.*

Theorem 1.2 *Every L-domain can be represented by a disjunctive system.*

To prove the first theorem a characterization of finite elements is needed. They turn out to be those which are generated by a *finite* set of tokens relative to a given element. The disjoint nature of the sequents makes it true that any *relative intersection* of a collection of compatible elements is again an element. Therefore, the greatest lower bound of any collection of elements exists within every principal ideal. In fact this is the construction with which we get the finite elements, as well as the least upper bounds of compatible sets.

To show that every L-domain can be represented by a disjunctive system, we use the notion of stable neighborhoods. Stable neighborhoods were introduced in [Zh89a] to give a topological characterization of stable functions. They are those Scott open sets whose minimal points are pairwise incompatible. The disjoint property of stable neighborhoods makes them natural candidates for propositions of disjunctive systems. We show that there is an order preserving, 1-1 correspondence, between an L-domain and the disjunctive system constructed from its stable neighborhoods.

Since disjunctive systems are based on a limited use of sets, they provide a concrete approach to domains. What is more important is, however, that they suggest an interesting classification of domains. Call a cpo an $[m, n]$-domain if it is determined by a disjunctive system whose sequents are of the form $X \vdash Y$ with $|X| \leq m$ *and* $|Y| \leq n$. Similarly, call a cpo an (m, n)-domain if it is determined by a disjunctive system whose sequents are of the form $X \vdash Y$ with $|X| < m$ *or* $|Y| < n$. Note that to be able to express disjointness, X should be allowed to have at least two elements. Note also that for technical advantage, X is required to be non-empty. For convenience let $*$ stand for *no restriction*.

By this classification, coherent spaces [Gi87] are exactly the [2, 0]-domains. Other existing domains can also be classified in this way. We summarize the classification by the following table.

Parameters	**Domain Types**
[2, 0]	Coherent Spaces
$[*, 1]$	Scott Domains
(2, 1)	Distributive Scott Domains
$(*, *)$	L-Domains

On the other hand, the classification of domains in this way brings forward an extremely rich family of domains. Many interesting issues arise. For example: What are the [3, 2]-domains? Do they form a cartesian closed category? Note these are follow-up questions which we do not attempt to settle here. Whether this classification generates any interesting category of domains or not remains to be seen.

The general notion of sequent structures and non-deterministic information systems were introduced in [Zh] and [DG90], respectively. The purpose of [Zh] was to give a representation of SFP domains (see also [Zh90b]), and the main purpose of [DG90] was to give a characterization of partial orders representable by non-deterministic information systems. This paper uses a different structure which is not contained in sequent structures or non-deterministic information systems: instead of having $\vdash$ as a relation between *finite subsets* of tokens, we need it to be a relation that relates a finite subset to *any* pairwise disjoint subset. As mentioned at the end, whether this need is absolutely necessary is unknown at the time this paper is finished.

2 L-Domains and Stable Neighborhoods

We assume the reader's familiarity with some basic definitions of domain theory, such as complete partial orders (cpos), finite elements, Scott open sets, algebraic cpos, etc. With respect to a cpo D, we write $\uparrow T$ for the upper closed set (upper set)

$$\{d \in D \mid d \sqsupseteq t \text{ for some } t \in T\},$$

where T is a subset of D. Similarly $\downarrow T$ stands for the set

$$\{d \in D \mid d \sqsubseteq t \text{ for some } t \in T\}.$$

In cases where $T = \{x\}$, a singleton set, we just write $\uparrow x$ or $\downarrow x$.

The notion of minimal upper bounds are quite relevant here, so we recall some related definitions. Let T be a subset of a cpo D. An element d is said to be a minimal upper bound of T if d dominates every element in T, and there is no element strictly below d with this property. The set of minimal upper bounds of T is written as $\bowtie T$. We call $\bowtie T$ complete if whenever d is an upper bound of T, d is already bigger than some element of $\bowtie T$. A cpo D is said to have *property m* if every subset has a complete set of minimal upper bounds.

For a given set T of a cpo, we define

$$\mathcal{U}^0(T) = T,$$
$$\mathcal{U}^{n+1}(T) = \{x \mid x \in \bowtie S \text{ for some finite subset } S \subseteq \mathcal{U}^n(T)\},$$
$$\mathcal{U}^\infty(T) = \bigcup_{n \geq 0} \mathcal{U}^n.$$

Let D be an algebraic cpo (the non-algebraic ones do not seem to be interesting). It is called an L-domain if for every x in D, the principal ideal $\downarrow x$ is a complete lattice. The following characterization of L-domains is useful.

Theorem 2.1 *(Jung) An algebraic cpo D is an L-domain if and only if D has property m and $\mathcal{U}^\infty = \mathcal{U}^1$.*

When we show that every L-domain can be represented as a disjunctive system later in Section 4, stable neighborhoods are used.

Definition 2.1 *Let D be a cpo. u is a stable neighborhood of D if it is a Scott-open set of D such that $u = \uparrow C$, where C is a collection of finite elements of D with the property*

$$(c_1, c_2 \in C \ \& \ \uparrow c_1 \cap \uparrow c_2 \neq \emptyset) \Longrightarrow c_1 = c_2.$$

We write the set of stable neighborhoods of a cpo D as $\mathbf{SN}(D)$. In general, $\mathbf{SN}(D)$ does not necessarily form a topology. It is not closed under finite intersections, neither arbitrary unions. However, stable neighborhoods are closed under *disjoint unions.*

Definition 2.2 *Let D be a cpo. The set of minimal points of a Scott open set u, written as μu, consists of $m \in u$ such that $\forall x \sqsubseteq m.\ x \in u \Longrightarrow x = m$.*

Clearly minimal points of a Scott open set are finite elements. Also, a Scott open set u is a stable neighborhood if and only if μu is pairwise incompatible. As a corollary of Theorem 2.1, we have a characterization of L-domains in terms of stable neighborhoods.

Proposition 2.1 *Let D be an algebraic cpo. D is an L-domain if and only if the collection of stable neighborhoods on D is closed under finite intersections.*

3 Disjunctive Systems

Definition 3.1 *A disjunctive system is a pair*

$$\underline{A} = (A, \vdash)$$

where A is a set of tokens and $\vdash$ is a relation such that whenever $X \vdash Y$, X is a non-empty finite subset of A and Y is a pairwise disjoint subset of A in the sense that

$$\forall a, b \in Y.\ a \neq b \Longrightarrow \{a, b\} \vdash \emptyset.$$

Note that the following additional axioms can be put on $\vdash$:

$$(Identity) \qquad a \vdash a,$$

$$(Weakening) \qquad \frac{X' \supseteq X \quad X \vdash Y \quad Y \subseteq Y' \quad Y \in \#A}{X' \vdash Y'},$$

$$(Cut) \qquad \frac{X \vdash Y, a \quad a, X' \vdash Y' \quad Y \cup Y' \in \#A}{X, X' \vdash Y, Y'}.$$

Here $\#A$ is the collection of pairwise disjoint subset of A.

When these axioms are satisfied we call a disjunctive system *normal*. However, as far as the representation of domains is concerned, these axioms may not be required. From the definition of elements given later, we will see that if any of the axioms is violated, the associated propositions can not appear in any element. In any case, every L-domain can be represented by a normal disjunctive system, as can be seen from the proof of Theorem 1.2. Additionally, it seems neater not to require normality when dealing with the classification of domains determined by disjunctive systems.

Note that when we write $X \vdash Y$, X is always a non-empty, finite subsets of propositions, and Y pairwise disjoint. Of course Y can be infinite, or even uncountable, keeping in mind that our intended purpose is to recast various kinds of domains in a general setting, including L-domains, which may have an uncountable number of finite elements.

A disjunctive system determines a family of subsets of propositions called its *elements.*

Definition 3.2 *The elements* $|\underline{A}|$ *, of a disjunctive system* $\underline{A} = (\, A, \vdash \,)$ *consist of subsets* x *of propositions which are closed under entailment:*

$$(X \subseteq x \;\&\; X \vdash Y) \Longrightarrow x \cap Y \neq \emptyset.$$

Given an element x of a disjunctive system $\underline{A}$, we have $|Y \cap x| = 1$ for any entailment $X \vdash Y$ with $X \subseteq x$. This follows from the pairwise disjoint nature of Y. Note for any disjunctive system $\underline{A}$, we have $\emptyset \in \mid \underline{A} \mid$. Although this is easy to check, we had to take some care earlier for the definition of the entailment relation. The requirement that X be non-empty in any entailment $X \vdash Y$ is crucial. Imagine if $\emptyset \vdash \emptyset$ were allowed in a disjunctive system $\underline{A}$. According to the definition of an element of a disjunctive system, $\mid \underline{A} \mid$ would be empty, contradicting the expectation that $(\mid \underline{A} \mid, \subseteq)$ be a cpo. This is not surprising because logically speaking, $\emptyset \vdash \emptyset$ means **true** $\vdash$ **false**, resulting in an inconsistent structure.

Noting that for disjunctive systems the least upper bound of a directed set is the union of elements in that set, we have

Proposition 3.1 *For a disjunctive system* $\underline{A}$*,*

$$(|\underline{A}|\,, \subseteq)$$

is a complete partial order.

The following proposition is the key to the construction of finite elements and the proof of the representation theorems.

Proposition 3.2 *Let* $\underline{A}$ *be a disjunctive system, and let* T *be a compatible subset of elements of* $\underline{A}$*. Then* $\bigcap T$ *is an element of* $\underline{A}$*.*

Proof Since T is compatible, there is some element x of $\underline{A}$ such that

$$\forall y \in T.\, y \subseteq x.$$

Let $X \vdash Y$ be an entailment in $\underline{A}$ such that $X \subseteq \bigcap T$. We have $X \subseteq y$ for every y in T. But such y's are elements of $\underline{A}$. Therefore $Y \cap y = Y \cap x \neq \emptyset$ for all y in T. This implies $Y \cap \bigcap T \neq \emptyset$.

$\Box$

Suppose P is any subset of an element x. According to the previous proposition,

$$\bigcap\{y \mid P \subseteq y \subseteq x \;\&\; y \in |\underline{A}|\}$$

is an element of $\underline{A}$. We write $\lceil P \rceil_x$ for such an element. Elements of the form $\lceil P \rceil_x$ with P finite are of special interest to us. They form the basis of a cpo $(|\underline{A}|\,,\,\subseteq)$.

Proposition 3.3 *Given a disjunctive system $\underline{A}$, $\lceil P \rceil_x$ is a finite element of $(|\underline{A}|\,,\,\subseteq)$ for any finite subset P of an element $x \in |\underline{A}|$.*

Proof Let S be a directed subset in the cpo $(|\underline{A}|\,,\,\subseteq)$, and let

$$\lceil P \rceil_x \subseteq \bigcup S.$$

(Note that $\bigsqcup S = \bigcup S$ when S is directed.) Clearly $\lceil P \rceil_x = \lceil P \rceil_{\bigcup S}$. Since $P \subseteq \bigcup S$, $P \subseteq z$ for some $z \in S$. Therefore

$$z \supseteq \lceil P \rceil_{\bigcup S} = \lceil P \rceil_x.$$

$\Box$

4 Representation Theorems

In this section we present the proofs of the two representation theorems mentioned in the introduction.

Proof of Theorem 1.1: Given a disjunctive system $\underline{A}$, and P, Q, subsets of an element x of the disjunctive system, we have

$$\lceil P \rceil_x \sqcup \lceil Q \rceil_x = \lceil P \cup Q \rceil_x.$$

It is then clear that

$$x = \bigcup \{ \lceil P \rceil_x \mid P \text{ is a finite subset of } x \},$$

with

$$\{ \lceil P \rceil_x \mid P \text{ is a finite subset of } x \}$$

directed. Hence $(|\underline{A}|, \subseteq)$ is algebraic. It is an L-domain because by Proposition 3.2, any subset S of a principle ideal $\downarrow x$ has a greatest lower bound, which is the intersection of all elements in S. Again by Proposition 3.2, the least upper bound of any subset S is the element

$$\bigcap \{ y \mid \bigcup S \subseteq y \subseteq x \;\&\; y \in |\underline{A}| \}.$$

Hence $\downarrow x$ is a complete lattice.

$\square$

Given an L-domain D, we can associate it with a disjunctive system

$$\mathcal{I}(D) = (A, \vdash).$$

Here A is the collection of stable neighborhoods of D excluding D, and $\vdash$ is given by

$$X \vdash Y \qquad \text{if} \qquad \bigcap X \subseteq \bigcup Y,$$

with X finite and Y pairwise disjoint (in set theoretic terms).

We now show that every L-domain can be represented as a disjunctive system.

Proof of Theorem 1.2: Let D be an L-domain. It is enough to prove that

$$D \cong |\mathcal{I}(D)|.$$

Define a mapping

$$\varphi:\ (D,\,\sqsubseteq)\rightarrow(|\underline{A}|\,,\ \subseteq)$$

by letting

$$\varphi(d)=\{u\mid u\in\mathbf{SN}(D)\setminus\{D\}\ \&\ d\in u\}.$$

We show that φ is an order preserving isomorphism between $(D,\,\sqsubseteq)$ and $(|\underline{A}|\,,\ \subseteq$ $)$.

The non-trivial part is to show that φ is onto. Let $x\in|\mathcal{I}(D)|$ be a non-bottom (otherwise $\varphi(\perp)=x$). We show that $\bigcap x=\uparrow t$ for some $t\in D$. For any $u\in x$, we have

$$\{u\}\vdash\{\uparrow p\mid p\in\mu u\}.$$

Therefore, there is some $p_0\in\mu u$ such that $\uparrow p_0\in x$, since x is an element of $\mathcal{I}(D)$. It follows that

$$\bigcap x=\bigcap\{\uparrow p\mid\uparrow p\in x\}.$$

However, $\{p\mid\uparrow p\in x\}$ is a directed set. To see this, let $p_0,\,p_1\in\{p\mid\uparrow p\in x\}$. The fact that D is an L-domain implies that $\bowtie\{p_0,\,p_1\}$ is a complete set of minimal upper bounds of $\{p_0,\,p_1\}$. Moreover, $\uparrow\bowtie\{p_0,p_1\}$ is again a stable neighborhood, by Proposition 2.1. Hence

$$\{\uparrow p_0,\,\uparrow p_1\}\vdash\{\uparrow p\mid p\in\bowtie\{p_0,\,p_1\}\}.$$

It follows that $\uparrow p\in x$ for some $p\in\bowtie\{p_0,\,p_1\}$. This p is an upper bound of p_0 and p_1.

Therefore

$$\bigcap x=\uparrow t,$$

where

$$t=\bigsqcup\{p\mid\uparrow p\in x\}.$$

From this we know that for any $u\in\mathbf{SN}(D)$ containing t, u must belong to x

since

$$\begin{aligned}
t \in u &\Longrightarrow (\bigsqcup\{p \mid \uparrow p \in x\}) \in u \\
&\Longrightarrow p_0 \in u \;\text{ for some }\; \uparrow p_0 \in x \\
&\Longrightarrow \uparrow p_0 \subseteq u \\
&\Longrightarrow \{\uparrow p_0\} \vdash \{u\} \\
&\Longrightarrow u \in x.
\end{aligned}$$

In summary, all the above shows that $x = \varphi(t)$ for some $t \in D$.

$\square$

5 Classification of Domains

A disjunctive system is called $[m,\, n]$ if all the sequents involved are of the form $X \vdash Y$ with $|X| \leq m$ *and* $|Y| \leq n$. It is $(m,\, n)$ if the sequents are of the form $X \vdash Y$ with $|X| < m$ *or* $|Y| < n$. The corresponding cpos are called $[m,\, n]$-domains or $(m,\, n)$-domains, respectively. To explain the notation better, note that $[m,\, n]$ suggests a 'closed interval', and $(m,\, n)$ an 'open interval'.

Consider a disjunctive systems $(A,\, \vdash)$, where all the entailments $X \vdash Y$ are such that $|X| = 2$ and $|Y| = 0$. Clearly $x \in |\underline{A}|$ if and only if

$$\forall a,\; b \in x.\; \{a,\; b\} \not\vdash \emptyset.$$

Therefore, the elements of such systems have the property that

$$(x \in |\underline{A}| \;\;\&\; y \subseteq x) \Longrightarrow y \in |\underline{A}|\,.$$

Hence every such a disjunctive system represents a coherent space. Conversely, it can be easily seen that every coherent space can be represented by a disjunctive system of this kind. We have, therefore,

Proposition 5.1 *The* $[2,\, 0]$*-domains are exactly coherent spaces.*

Implied by the results of Scott [Sc82], we have

Proposition 5.2 *The* $[*, 1]$*-domains are exactly Scott domains.*

Suppose $(A, \vdash)$ is a disjunctive system with the entailments of the form $X \vdash Y$ such that either $|X| = 1$ or $|Y| = 0$. What can be said about the cpo it represents? It is a distributive Scott domain. Let T be a compatible subset of $|\underline{A}|$. It can be seen that the union $\bigcup T$ must also be an element. Therefore the elements ordered under inclusion is a Scott domain. It is distributive because in this case the greatest lower bound of any collection of elements is just the intersection of these elements.

Further more, any distributive Scott domain can be represented by a disjunctive system of this kind, even with the condition $|Y| \leq 1$ holding for all entailments $X \vdash Y$. The proof of this claim can be derived from [Zh89]. This leads to the following proposition.

Proposition 5.3 *The* $(2, 1)$*-domains are exactly distributive Scott domains.*

Continuing in this line, we only need that the following relation $\prec$ be well-founded for a disjunctive system $(A, \vdash)$ to represent a dI-domain. For $a, b \in A$, $a \prec b$ if $a \neq b$ and for some Y containing b, $a \vdash Y$. The well-foundedness of $\prec$, however, does not seem to be expressible in terms of the types of disjunctive systems as it stands: I suspect dI-domains are none of the $[m, n]$-domains or (m, n)-domains.

6 Categorical Equivalence

This section introduces approximable mappings between disjunctive systems. It then shows that the category of disjunctive systems with approximable mappings is equivalent to the category of L-domains.

For technical convenience we work with a particular kind of disjunctive systems — those which are normal (see Section 3) and *expressive*.

Definition 6.1 *A normal disjunctive system* $(A, \vdash)$ *is called expressive if*

$$\{a\} \vdash Y \Longrightarrow \exists b \in Y.\ \{a\} \vdash \{b\},$$

and for every finite subset $X \subseteq A$*, there is some* $Y \subseteq A$ *such that*

$$X \vdash Y \ \&\ \forall b \in Y.\ [\forall a \in X.\ \{b\} \vdash \{a\}].$$

We abbreviate the above statement as $X \rightharpoonup Y$ *(note* $\rightharpoonup$ *is not symmetric).*

Note that from Proposition 2.1 and the proof of Theorem 1.2 we know that the expressiveness condition will not pose a restriction on the expressive power of disjunctive systems. This will be confirmed again later in the categorical equivalence theorem.

We now introduce morphisms on expressive disjunctive systems called approximable mappings. This makes expressive disjunctive systems a category. Approximable mappings show how disjunctive systems are related to one another and they correspond to continuous functions between the associated L-domains.

Definition 6.2 *Let* $\underline{A} = (\,A, \vdash_{\underline{A}}\,)$*,* $\underline{B} = (\,B, \vdash_{\underline{B}}\,)$ *be expressive disjunctive systems. An approximable mapping from* $\underline{A}$ *to* $\underline{B}$ *is a relation* $R \subseteq A \times B$ *which satisfies*

$$\begin{array}{l} \forall S \subseteq^{fin} R \quad \forall X \forall Y. \\ \qquad\qquad (\pi_1 S \rightharpoonup_{\underline{A}} X \ \&\ \pi_2 S \rightharpoonup_{\underline{B}} Y) \Longrightarrow \forall a \in X \exists b \in Y.\ a\,R\,b. \end{array}$$

Here π_1 and π_2 are projections to the first and the second component, respectively, and $\subseteq^{fin}$ stands for 'a finite subset of'.

Clearly, when $a \vdash a'$ we have $a' \rightharpoonup \{\,a,\, a'\,\}$. Taking $\{(a',\, b')\} \subseteq R$, $X = \{\,a,\, a'\,\}$, $Y = \{b'\}$ in the previous definition, we have, for an approximable mapping R,

$$a \vdash a' \ \&\ a'\,R\,b' \Longrightarrow a\,R\,b'.$$

Proposition 6.1 *Expressive disjunctive systems with approximable mappings form a category, written as* **EDIS**.

Proof We check that approximable mappings compose. Other axioms for a category can be checked similarly. Let $\underline{A}$, $\underline{B}$ and $\underline{C}$ be expressive disjunctive systems and $R : \underline{A} \to \underline{B}$ and $S : \underline{B} \to \underline{C}$ be approximable mappings. Let $R \circ S$ be the relational composition. We show that $R \circ S$ is an approximable mapping. Suppose, for a finite set I, $\forall i \in I.\ a_i\,(R \circ S)\,c_i$ and $\{\, a_i \mid i \in I \,\} \rightharpoonup_{\underline{A}} X$, $\{\, c_i \mid i \in I \,\} \rightharpoonup_{\underline{C}} Z$. There exists a $u_i \in B$ such that $a_i\,R\,u_i$, $u_i\,S\,c_i$ for any $i \in I$. Let $\{\, u_i \mid i \in I \,\} \rightharpoonup_{\underline{B}} Y$. The existence of such Y follows from the expressiveness of $\underline{B}$. Since R is an approximable mapping, $\forall p \in X \exists q \in Y.\ p\,R\,q$. But for each $q \in Y$ we have $q\,S\,c_i$ for all $i \in I$. Therefore there exists some $r \in Z$ such that $q\,S\,r$, since $q \rightharpoonup q$. Hence $\forall p \in X \exists r \in Z.\ p\,(R \circ S)\,r$.

$\square$

Approximable mappings determine continuous functions via the construction given in the following definition.

Proposition 6.2 *Let R be an approximable mapping from $\underline{A}$ to $\underline{B}$. Define $f_R : |\underline{A}| \to |\underline{B}|$ by*

$$f_R(x) = \{\, b \in B \mid \exists a \in x\,.\ a\,R\,b \,\}.$$

Then f_R is a continuous function from $|\underline{A}|$ to $|\underline{B}|$.

Proof The only difficulty is to show that f_R is well-defined. Let $x \in |\underline{A}|$ and let $R : \underline{A} \to \underline{B}$ be an approximable mapping. To show $f_R(\,x\,) \in |\underline{B}|$ let $Y \subseteq^{fin} f_R(\,x\,)$ and $Y \vdash_{\underline{B}} Z$. For each $b \in Y$ there is some $a \in x$ such that $a\,R\,b$. Write X for such a collection of a's. Because $\underline{A}$ and $\underline{B}$ are expressive, there are X', Y' such that $X \rightharpoonup_{\underline{A}} X'$ and $Y \rightharpoonup_{\underline{B}} Y'$. This means we have $X \subseteq x$ and $X \vdash_{\underline{A}} X'$, which implies $X' \cap x \neq \emptyset$. Now let $u_0 \in X' \cap x$. By Definition 6.2 there is a $v_0 \in Y'$ such that $u_0\,R\,v_0$. Thus $v_0 \in f_R(x)$. We must also have $v_0 \vdash_{\underline{B}} Z$, which implies, by expressiveness, $v_0 \vdash_{\underline{B}} c$ for some $c \in Z$. Therefore $c \in f_R(\,x\,) \cap Z$. Thus f_R is well-defined.

$\square$

To show that the category of expressive disjunctive systems and the category of L-domains (which is written as $\mathbf{L}$) are equivalent, we use one of MacLane's results ([Ma71], pp 91). By this result, a functor F determines an equivalence of the categories if it is full and faithful, and each L-domain D is isomorphic to $F(\underline{A})$ for some expressive disjunctive system $\underline{A}$.

Theorem 6.1 **EDIS** *is equivalent to* $\mathbf{L}$.

Proof Let $F : \mathbf{EDIS} \to \mathbf{L}$ be the functor given by

$$F\,(\,\underline{A}\,) = |\underline{A}|$$
$$F\,(\,R\,) = f_R.$$

That each L-domain D is isomorphic to $F(\underline{A})$ for some expressive disjunctive system $\underline{A}$ is shown in Theorem 1.2. It remains to show that F is full and faithful. First we show that F is full. Let $\underline{A}$ and $\underline{B}$ be expressive disjunctive systems, and $f : F(\underline{A}) \to F(\underline{B})$ a continuous function. Define a relation $R \subseteq A \times B$ by letting $a\,R\,b$ if $b \in f(\overline{a})$. We check that this relation is an approximable mapping form $\underline{A}$ to $\underline{B}$. Let $\{\,(a_i,\, b_i) \mid i \in I\,\}$ be a finite subset of R. Assume

$$\{\,a_i \mid i \in I\,\} \rightharpoonup_{\underline{A}} X$$

and

$$\{\,b_i \mid i \in I\,\} \rightharpoonup_{\underline{B}} Y.$$

For any $a \in X$, we have $a \vdash \{a_i\}$ for every $i \in I$. Thus we have $b_i \in f(\overline{a_i}) \subseteq f(\overline{a})$ for any $i \in I$. Now $\{\,b_i \mid i \in I\,\} \vdash_{\underline{B}} Y$. Therefore $f(\overline{a}) \cap Y \neq \emptyset$. This means for some $b \in Y$, $b \in f(\overline{a})$, or $a\,R\,b$.

We now show that the continuous function f_R determined by the above R is actually equal to f. Let $x \in |\underline{A}|$. Suppose $b \in f_R(x)$. By definition there is some $a \in x$, $a\,R\,b$. That is, $b \in f(\overline{a})$. Therefore $b \in f(x)$, by the monotonicity of f. Thus $f_R(x) \subseteq f(x)$. On the other hand, let $b \in f(x)$. By the continuity of f there is some $a \in x$ such that $b \in f(\overline{a})$. Hence $a\,R\,b$ and $b \in f_R(x)$. This means $f(x) \subseteq f_R(x)$. Hence $f = f_R$.

Secondly, we show that F is faithful. Suppose $R, S : \underline{A} \to \underline{B}$ are approximable mappings such that $f_R = f_S$. Let $a\,R\,b$. Then $b \in f_S(\overline{a})$. This means for some $a' \in \overline{a}$, $a'\,S\,b$, which implies $a\,S\,b$. Therefore, $R \subseteq S$. By symmetry, $S \subseteq R$ and hence $R = S$.

□

7 Future Work

There are many interesting issues to be explored about disjunctive systems.

A sub-system relation can be introduced to disjunctive systems using the method presented in [LaWi84], so that recursive equations can be solved by using fixed-point constructions. Technically, we do not anticipate any difficulty for such a treatment. The question is, however: what kind of equations do we have? The equations usually involve constructions on disjunctive systems. However, we have not dealt with constructions such as function space on the category of expressive disjunctive systems. It is not clear to the author at this moment how best function space can be introduced (that is the reason we have not included a sub-system relation).

Secondly, there is the question of whether $(*, *)$-domains and $(*, \infty)$-domains are the same (i.e., L-domains). On the surface, it looks like $(*, \infty)$-domains should be the bifinite L-domains. Clearly, every bifinite L-domain can be represented by a $(*, \infty)$-disjunctive system. The other way round, however, is not true: $(*, \infty)$-disjunctive systems can represent more than just the bifinite L-domains. Here is an example.

Example (Droste and Gobel) *Let* $(A, \vdash)$ *be a disjunctive system with*

$$A = \{a, b\} \cup \{1, 2, 3, \cdots\}$$

and the entailment relation given by

$$\{a,\ b\} \vdash \{1,\ 2\},$$

$$\{i\} \vdash \{a\},\ \ \{i\} \vdash \{b\}\ \ \textit{for all } i \geq 1,$$

$$\{2i+1\} \vdash \{2i\},\ \ \{2i+2\} \vdash \{2i\}\ \textit{for all } i \geq 1,$$

$$\{2i\} \vdash \{2i+1,\ 2i+2\}\ \ \textit{for all } i \geq 1,\ \textit{and}$$

$$\{2i-1,\ 2i\} \vdash \emptyset\ \ \textit{for all } i \geq 1.$$

The elements determined by this disjunctive system are:

$$\emptyset,\ \{a\},\ \{b\}\ \text{ and}$$

$$\bar{i} = \{a,\ b\} \cup \{2,\ 4,\ \cdots,,2i-2\} \cup \{2i-1\} \text{ for } i \geq 1, \ \text{ as well as}$$

$$\overline{\infty} = \{a,\ b\} \cup \{2i \mid i \geq 1\}.$$

Ordered by set inclusion we get a cpo which is not bifinite. One can even have an uncountable set of minimal upper bounds for a finite set of finite elements in a $(*,\ \infty)$-domain. Given *any* non-empty set Q, one can construct the following $(*,\ \infty)$-disjunctive system. Let the token set be

$$\{a,\ b\} \cup Q \times \{0\} \cup Q \times \{1\}.$$

The entailment relation is specified by

$$\{a,\ b\} \vdash \{(q,\ 0),\ (q,\ 1)\}\ \text{ for all } q \in Q,$$

$$\{(q,\ 0),\ (q,\ 1)\} \vdash \emptyset\ \text{ for all } q \in Q.$$

There is a 1-1 correspondence between functions

$$f:\ Q \to \{0,\ 1\}$$

and elements

$$\{a,\ b\} \cup \{(q,\ f(q)) \mid q \in Q\}.$$

Therefore, elements $\{a\}$, $\{b\}$ have a set of minimal upper bounds with cardinality the same as that of the power set of Q.

All this suggests that maybe $(*, \infty)$-domains are L-domains. We have no proof of this at the moment: the proof of Theorem 1.2 cannot be easily adapted to the new situation. It seems some intrinsic construction must be used to get a $(*, \infty)$-disjunctive system for an L-domain. However, it would be even more interesting if one could find an example of an L-domain which is not $(*, \infty)$. What, then, are the $(*, \infty)$-domains?

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