DOMAIN THEORY MEETS DEFAULT LOGIC

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Abstract. We present a development of the theory of default information structures, combining ideas from domain theory with ideas from non-monotonic logic. Conceptually, our treatment is distinguished from standard default logic in that we view default structures as generating models rather than theories. Reiter's default rules are viewed as non-deterministic algorithms for generating preferred partial models. Using domain-theoretical notions, we improve the standard definition of extensions in default logic, by introducing the notion of *dilation.* We prove the existence of such dilations for a new, natural class of default information structures, properly including the so-called semi-normal ones. This class, called the class of *rational* structures, is a robust generalization of the usual kind of default rule system.

Key words: domain theory, non-monotonic reasoning, default logic, sequent structures, information systems.

1 Introduction

1.1 Background and related work

Default logics are a class of non-monotonic logics, formal systems for reasoning about common sense and beliefs. An important characteristic of such logics is that conclusions made from beliefs may have to be withdrawn when new evidence disproves them.

Default logic is a generalization of first order logic with proof rules, called default rules, added to the system to model the consequences of hypothetical assumptions. The sets of such consequences are called extensions. When engaged in preliminary work on the paper, we noticed that the notion of extension suffers from certain fragility problems, and we wondered if there were not a more robust alternative. What follows is an outline of our proposal.

Perhaps the most basic observation is there has been too much dependence in the world of non-monotonic logic on the very idea of a logic. We think that the notion of formal proof is probably not the right way to model the kind of non-monotonic reasoning done by humans. Most researchers probably suspect that real reasoning is semantic. We suggest that instead of thinking about formal proofs, one should consider the more general notion of *information*, a move suggested by the theory of denotational semantics of programming languages, and also by situation theory [3].

Toward this end, we introduce the concept of *default information structures*. These extend the notion of Scott's *information systems* [18], a general framework accounting for semantic information in the monotonic world. Information systems give an exact characterization of *Scott domains*, which are certain kinds of complete algebraic partial orders. We extend the idea to the non-monotonic world, combining ideas from default logic and ideas from domain theory. We think that the generalization opens up a host of new possibilities for investigation. Among other things, we find that default information structures determine partial orders which are not Scott domains, and not even cpo's.

The primary notions in default logic are those of a default theory and its extensions. A default theory consists of a pair $(T,\ \Delta)$, where T is a set of first order formulae, Δ is a collection of *default rules* of the form

$$\frac{\alpha_1,\ \alpha_2, \cdots\ \alpha_m : \beta_1,\ \beta_2,\ \cdots,\ \beta_n}{\gamma}.$$

The intuitive meaning is that if the α_i's are part of our current beliefs, and none of the $\neg\beta_j$'s are current, then it is allowable to believe γ.

Informally, an extension of such a theory is a minimal collection of formulae that can be derived from T by using default rules in Δ but invariant under applications of such rules. There are certain difficulties with the notion of extension – primarily, that they do not exist when they seemingly should. We therefore replace extensions with a new notion, called *dilations*. Dilations are a weakening of the idea of extension. We give the formal definition in Section 3, but some discussion is in order now.

We quote Reiter [15, sec. 1.3] on the subject of extensions:

> The role of a default is to fill in some of the gaps in the knowledge base ... so as to permit the inferences necessary to act. Defaults therefore function somwewhat like meta-rules; they are instructions about how to complete an *extension* of this incomplete theory ... Now in general there may be many ways of extending an incomplete theory, which suggests that the default rules may be nondeterministic. Different applications of the defaults yield different extensions and therefore different sets of beliefs about the world.

The concept of dilation is in the exact spirit of Reiter's discussion. (It is clear that by "extension" Reiter means "an extended set of beliefs" and not the logician's notion of extension as distinct from intension.) As Reiter does, we make a distinction between established facts, and things we believe, including those derived from default rules. We express this using a *dilation pair* $x\delta\ y$, where x represents *established* information, and y represents *plausible* information. x and y are just sets of pieces of information, and of course if $x\delta\ y$ (read y is a dilation of x) then $x \subseteq y$. The difference between the dilation relation and the extension relation is in the way defaults are employed to build the larger world y. Here Reiter's intuitive view of default rules as *instructions* is very important. In building a dilation, we try not to think of default rules as proof rules. Our world-building algorithms for dilations are quite removed from the notion of proof.

World building is modeled in two ways in our structures. We have a relation between pieces of information (usually notated $\vdash$) which models monotonic (strict) forcing of pieces of information, and we have a consistency (better: coherence) predicate on worlds (in fact, this can be derived from the proper notion of entailment). Then, we use *defaults*, of just the same form as above, to mean that *if α_i's are currently plausible, and all the β_j's are consistent with everything currently plausible, then γ is also plausible.*

One difference between our systems and default theories is that we separate strict from default inference. Another, more fundamental, difference is that *we do not specify beforehand what pieces of information are.* These pieces are just tokens; the only constraint on them is given by meta-axioms on the relation $\vdash$, and also on the default relations. It is this fundamental difference which gives our approach its robustness. Of course, the tool has already proved its worth in the standard theory of domains.

Among the papers on default systems in the current literature, one which seems to use methods most closely allied to ours is a paper of Marek, Nerode, and Remmel (MNR) [12]. Inspired by Reiter's original work [15], and by that of Apt [2], MNR generalize default systems using "tokens" in the way we have outlined. They do not assume that a logical language is pre- specified. Instead, they work with structures of the form

$$\langle U, N \rangle$$

where U is an unstructured set of "objects", and N is a set of default rules somewhat like those we have shown above. They show that their systems include many of the systems in the literature. Interesting algebraic and combinatorial problems can be analyzed in terms of extensions, where the objects represent things like pairings of boys and girls, and extensions represent solutions to the stable marriage problem. Finally, they present a comparison of some of their work to constructions in logic programming, and analyze the recursion-theoretic character of extensions.

Our paper is in the same spirit as that of MNR, but we suggest some new considerations. Instead of working with an unstructured set U of tokens, we use Scott's fundamental insight that a *consistency* predicate be assumed to exist on (finite subsets of) U, and that a natural notion of *entailment* be defined as a relation between finite subsets of U as well. This allows us to divorce strict (monotonic) entailment from default entailment, and it gives us a natural setting to treat states of information, extensions, and dilations as fixed points. It also allows us to use compactness considerations in our results, as the notion of *ideal element* in domain theory presumes a compactness property.

As a short example of the advantages of assuming a consistency predicate, consider the basic type of default rule used in the Theorist system [13]:

$$\frac{: a}{a}.$$

The meaning is very clear: if the fact a is consistent, then believe it. The stable marriage problem is easily phrased this way, for example: we can use rules of the above form, where a is the fact that a boy is married to a girl he knows. On the side, using strict entailment, we can specify the *inconsistency* of a boy's being married to more than one girl at a time. Perhaps a more telling point is that in the framework of MNR, the meaning of the above rule is difficult to express.

Finally, we have found an application for our notion of default models. This involves a family of modal logics which use default information systems as their semantics. This work will appear shortly [17] .

We proceed with a review of some problems with default logic.

1.2 Faults of Default Logic

The default logic notion of extension is intended to represent the full consequences of our beliefs. However, in conventional default logic the existence of extensions is not guaranteed, even in reasonable situations where they should exist. There are other limitations of the conventional theory of default logic, which we discuss now.

1. *Default logic is not general enough.* Because the conventional theory of default logic is strongly based on the syntax of first order logic, it is not adequate to deal with beliefs in general, not to mention information, where negation may not make good sense. Consider the following beliefs, for an example:

(α) The chicken comes first.
(β) The egg comes first.

Represented as default rules, we have

$$\frac{: \alpha}{\alpha}, \quad \frac{: \beta}{\beta}.$$

Here, α and β are not the negation of one another; they are merely inconsistent with each other. Introducing negation here seems to be superficial. See the example following the definition of information systems in Section 2 for more evidence.

2. *Extensions do not exist in many reasonable cases.* Note that default logic has its excuse for this. It offers one of the possible ways to understand defaults and beliefs. It views conclusions derived from beliefs as evidence which may block the application of other default rules. However, we think that "extensions" should exist more often.

3. *Default logic offers a static, rather than a dynamic account for information.* The conventional theory of default logic deals with one theory at a time, and hence does not provide an account of the evolution of information. (We recognize that researchers in the semantics of beliefs have considered the dynamics of epistemic states, in the form of theories of belief revision and theory change (e.g. [11]). Default logic work, though, has not had a similar emphasis.)

4. *Default logic is proof-oriented.* It is this obstacle which is most difficult to remove. We want to espouse the view of default rules as *algorithms* for building belief spaces. The conclusion of a default rule should not be thought of as a formula, but as a piece of information. A belief space (of which dilations are one type) will consist of a coherent collection of such pieces of information. As an example, think of a belief space as a hypothetical spread sheet, which fills in incomplete spaces in a real one. (Specifically, a tax form in which wages have been estimated.) Coherence, or consistency, means that the spread sheet obeys relationships such as "tax is a certain function of adjusted gross income." The pieces of information involved here are just the cells of the spread sheet. Now, we think of strict information – the first component of a dilation pair – as the incomplete spread sheet without the wages. The second component – the dilation – is the spread sheet with the wages filled in. Obviously, there is no reason to suppose that this piece of information is filled in as the result of a proof. It might be filled in by a rule of thumb – take the state income taxes withheld and divide by the state tax rate.

Our framework helps us to overcome the above problems, especially (2) and (4). Information systems are the way to get at (4), and dilations help us with (2) because they allow us to have more consequences of our beliefs than before. This is because, in our framework, anything derived rationally from a consistent state of belief cannot serve as evidence based on which we withdraw conclusions already made, as long as our new set of beliefs is consistent.

Here is an example to clarify this point (for another more detailed example, see Section 5). In this case we work with standard default logic. Consider the default theory

$$(\emptyset,\ \{\frac{:a}{\neg a}\}).$$

According to conventional default logic, this theory has no extension. In our framework, it will turn out that

$$(\emptyset,\ Th(\{\neg a\})\ )$$

is a dilation pair (that is, an ordered pair in the dilation relation), where Th stands for 'standard consequences of'. (See the discussion on the first page for informal review of dilations, Section 3 for the formal definition, and section 5 for another example.) The intuitive reason that $Th(\{\neg a\})$ is not an extension is because $\neg a$ "essentially blocks" the application of the default rule. The same theory *is* a dilation of $\emptyset$, because the fact that $\neg a$ is a plausible consequence of $\emptyset$ allows us to show that $Th(\{\neg a\})$ is merely closed under the usual entailment rules and under the default rule.

1.3 Organization and content of this paper

Section 2 contains background material on information systems and sequent structures. These latter systems, fully explicated in Zhang [19], are a generalization of information systems to account for *nondeterministic* information, but still monotonic. Using the notations of sequent structures, we define default information structures in Section 3.

Section 4 recasts the definition of extension in first-order default logic in terms of information; this is in preparation for the more robust notion of dilation to follow. Two definitions of extensions as fixed points of certain operators are given. These lead us to generalize a characterization of Reiter [15], who shows that these two operators, in the setting of default logic, have the same fixed points. We show a much stronger result – in the setting of information systems, these operators are identical.

Section 5 is devoted to a study of our generalization of the notion of extension – the concept of dilation. As indicated above, the dilation relation is a set of pairs (x, y), where x is a set of "known" items of information, and y is a consistent set of "plausible consequences" of x. We show that for a large class of default information structures, properly including the so-called semi-normal ones, that for any consistent set x of known pieces of information, there is a dilation pair (x, y), where $x \subseteq y$. This class is the class of *rational* default structures, which we suggest as a model-theoretic basis for default reasoning. We show that for every rational default structure, there is a semi-normal one with the same dilation pairs. (This involves a crucial use of compactness.) Finally, we prove the fundamental result that *for semi-normal structures, and thus for rational ones, dilations always exist.* Such is not the case for extensions; moreover, the proof shows the existence of *maximal* dilations, invoking Zorn's lemma in the general case of arbitrary sets of tokens.

In the remainder of the paper, we consider only the class of rational default structures. Section 6 compares the notions of dilations and extensions. We show that for every consistent x, and for every extension y of x, that y is a minimal dilation extending x. But there can be minimal dilations, in finite examples, where there are no extensions. We show that even for normal default structures, extensions and dilations can be different. There is, however, one case in which they are the same: the class of *precondition-free* default structures, where the default relations are of the form

$$\frac{: a}{a}.$$

We also consider the relationship between dilations and the so-called em modified extensions of Lukaszewicz[9]. In this case there are no inclusions between dilations and modified extensions.

Section 7 contains some preliminary results on the class of partial orders determined by rational default information structures. It is shown that every such order is *upper-complete* – every nonempty upper directed set has a least upper bound. In contrast to most examples from denotational semantics, there can be infinite descending chains with no greatest lower bound. Furthermore, there can be an uncountable number of minimal elements, even for a countable and very recursively presented information structure. Even in the finite case, we may not have a Scott domain. Finally, we expect that the recursion-theoretic properties of dilations will be at least as complex as those considered by Marek, Nerode, and Remmel. But we have to leave this for further investigation.

2 Sequent Structures

In this section we recall the basic definitions of the theory of information systems and sequent structures.

Definition 2.1 *A sequent structure is a tuple*

$$\underline{A} = (A, \vdash)$$

where A is a set of tokens, and $\vdash$ is a relation between finite subsets of A, called the entailment relation, satisfying the following conditions:

$$\begin{array}{ll} (Identity) & a \vdash a, \\ (Weakening) & \dfrac{X' \supseteq X \quad X \vdash Y \quad Y \subseteq Y'}{X' \vdash Y'}, \\ (Cut) & \dfrac{X \vdash Y, a \quad a, X' \vdash Y'}{X, X' \vdash Y, Y'}. \end{array}$$

In addition, we require that $\emptyset \not\vdash \emptyset$.

It should be emphasized that the set A of tokens is analogous to a set of well-formed sentences, but need not actually be such a set. Nevertheless, we proceed by analogy to the logical case. Thus, when we write $X \vdash Y$ we are imagining that the conjunction of the elements of X implies the disjunction of the elements of Y. Moreover, by $X, X' \vdash Y, Y'$ we mean $X \cup X' \vdash Y \cup Y'$.

It is important to remember that when we write $X \vdash Y$, X and Y are always finite sets of tokens. Note also that since we do not actually have logical connectives and their related rules, *Cut* cannot be eliminated. In fact, *Cut* is necessary for the following theorem.

Theorem 2.1 *Let $(A, \vdash)$ be a sequent structure. Then*

1. $a \in X \Longrightarrow X \vdash a,$
   $a \in X \Longrightarrow a \vdash X,$
2. $(\,(\forall b \in Y.\, X \vdash b\,)\ \&\ Y \vdash Z\,) \Longrightarrow X \vdash Z,$
3. $(\ X \vdash Y\ \&\ Y' \subseteq Y\ \&\ \forall b \in Y'.\, b \vdash Z\ ) \Longrightarrow X \vdash (Y \setminus Y'),\, Z,$
4. $(\forall b \in Y.\ X,\, b \vdash \emptyset)\ \&\ X \vdash Y \Longrightarrow X \vdash \emptyset,$
5. $(\forall b \in Y.\ X \vdash b)\ \&\ X,\, Y \vdash \emptyset \Longrightarrow X \vdash \emptyset.$

This theorem remains true if $\emptyset$ is replaced by an arbitrary (finite, though) Z in items 4 and 5. However, it is these special forms that are needed in the proof of Theorem 2.2, showing that the axioms of information systems can be derived from the general properties of sequent structures.

Proof As an example, we prove the second conclusion in detail. This can be done by repeated applications of the cut rule. When Y is a singleton the conclusion follows by an application of the cut rule. In general, let $b \in Y$. We have $X \vdash b$ and $b, (Y \setminus \{\,b\,\}) \vdash Z$. Applying the cut rule we get $X, (Y \setminus \{\,b\,\}) \vdash Z$. Now choose a $b' \in (Y \setminus \{\,b\,\})$, if possible. We have $X \vdash b'$ and $b', (Y \setminus \{\,b, b'\,\}), X \vdash Z$. Applying the cut rule again, we get $X, (Y \setminus \{\,b, b'\,\}) \vdash Z$. Repeating this process a number of times until $Y \setminus \{\,b, b', b'', \cdots\}$ is empty, we get $X \vdash Z$.

□

A sequent structure determines a family of sets of tokens called its *ideal elements*. Occasionally, we omit the word "ideal" and simply speak of the elements of a structure.

Definition 2.2 *The ideal elements $|\underline{A}|$, of a sequent structure $\underline{A} = (\,A,\ \vdash\,)$ are subsets x of A which are closed under entailment:*

$$(X \subseteq x\ \&\ X \vdash Y) \Longrightarrow x \cap Y \neq \emptyset.$$

Now we recall the notion of information system. This is a structure originally used to describe the logical relations among propositions that can be made about computation. It consists of a set of information tokens, an entailment relation like that specified in Definition 2.1, and an extra consistency predicate. We use the definition given in Larsen and Winskel [8], which differs slightly from the original one given by Scott.

Definition 2.3 *An information system is a structure $\underline{A} = (\,A,\ Con,\ \vdash\,)$ where*

- A *is a set of tokens,*
- $Con \subseteq \mathrm{Fin}\,(A)$, *the consistent sets,*
- $\vdash \subseteq Con \times A$, *the entailment relation,*

which satisfy

1. $X \subseteq Y \;\&\; Y \in \mathit{Con} \Longrightarrow X \in \mathit{Con}$,
2. $a \in A \Longrightarrow \{\, a \,\} \in \mathit{Con}$,
3. $X \vdash a \;\&\; X \in \mathit{Con} \Longrightarrow X \cup \{\, a \,\} \in \mathit{Con}$,
4. $a \in X \;\&\; X \in \mathit{Con} \Longrightarrow X \vdash a$,
5. $(\, \forall b \in Y. X \vdash b \;\&\; Y \vdash c \,) \Longrightarrow X \vdash c$.

Example: Approximate real numbers. For tokens, take the set A of pairs of rationals $\langle q, r \rangle$, with $q \leq r$.

The idea is that a pair of rationals stands for the "proposition" that a yet to be determined *real number* is in the interval $[q, r]$ whose endpoints are given by the pair.

Define a finite set X of "intervals" to be in in *Con* if X is empty, or if the intersection of the "intervals" in X is nonempty. Then say that a set $X \vdash \langle q, r \rangle$ iff the intersection of all "intervals" in X is contained in the interval $[q, r]$. Note that there is only atomic structure to these propositions. We cannot negate them or disjoin them. So although tokens can conveniently be exemplified as logical formulas, they cannot always be assumed to be closed under the usual logical syntax. It is also clear that the $\vdash$ relation in general obeys laws like that of logical consequence, but is very much more general. We will take advantage of the generality in our definition of dilation, in which we will employ a "plausible entailment" operator which we do not intend to be equivalent to any "preferential consequence" logical operator.

An information system determines a family of ideal elements, in the same way a sequent structure does. In the following, we write $X \subseteq^{fin} y$ to mean X is a finite subset of y.

Definition 2.4 *The elements $|\underline{A}|$, of an information system $\underline{A} = (\, A,\ \mathit{Con},\ \vdash \,)$ consists of subsets x of propositions which are*

1. *finitely consistent:* $X \subseteq^{fin} x \Longrightarrow X \in \mathit{Con}$,
2. *closed under entailment:* $X \subseteq x \;\&\; X \vdash a \Longrightarrow a \in x$.

Example. The ideal elements in our approximate real system are in 1-1 correspondence with the collection of closed real intervals $[x, y]$ with $x \leq y$. Although the collection of ideal elements is partially ordered by inclusion, the domain being described – intervals of reals – is partially ordered by reverse interval inclusion. The total or maximal elements in the domain correspond to "perfect" reals $[x, x]$. The bottom element is a special interval $(-\infty, \infty)$.

We can show that information systems are a special class of sequent structures.

Definition 2.5 *A sequent structure $\underline{A}$ is called deterministic if it satisfies the following:*

1 $(X \vdash Y \& X \not\vdash \emptyset) \Longrightarrow \exists b \in Y \,.\, X \vdash b$,
2 $\forall a \in A.\; a \not\vdash \emptyset$.

In other words, a deterministic sequent structure is a sequent structure with a generating relation of the form $X \vdash Y$ with $|\, Y \,| \leq 1$ and, moreover, such that each singleton token is consistent.

Theorem 2.2 *For every deterministic sequent structure there is an information system with the same ideal elements.*

Proof Let $(A, \vdash)$ be a deterministic sequent structure. We can construct an information system in the following way. The information system has the same token set A. The entailment relation $\vdash'$ is the generating relation for $\vdash$. Define *Con* to be the set $\{\, X \mid X \not\vdash \emptyset\}$. We show that the five axioms for an information system are derivable from the sequent structure. It is sufficient to show that

1. $X \subseteq Y \;\&\; Y \not\vdash' \emptyset \Longrightarrow X \not\vdash' \emptyset$,
2. $a \not\vdash' \emptyset$,
3. $X \vdash' a \;\&\; X \not\vdash' \emptyset \Longrightarrow X,\, a \not\vdash \emptyset$,
4. $a \in X \Longrightarrow X \vdash' a$,
5. $(\,\forall b \in Y.\, X \vdash' b \;\&\; Y \vdash' c\,) \Longrightarrow X \vdash' c$.

It is easy to see that these follow from Theorem 2.1.

$\square$

The collection of ideal elements of a given information system forms a partial order under set inclusion. Scott's fundamental result is that the class of partial orders determined by information systems coincides with the abstract notion of *Scott domains*: consistently complete algebraic partial orders.

3 Default Information Structures

With the notation of information systems and sequent structures fixed, we come to the main definition of the paper.

Definition 3.1 *A default information structure is a triple*

$$\underline{A} = (A,\, \Delta,\, \vdash)$$

where A is a set of tokens, Δ is a ternary relation finite subsets of A, written

$$\frac{X : Y}{Z},$$

and $(A, \vdash)$ is a deterministic sequent structure.

Each element of Δ will be called a *default.* If

$$\frac{X : Y}{Z}$$

is a default, we read that fact as follows: if, according to our current information, the pieces of information (i.e. tokens) in X are plausible, and our current information together with the pieces of information in Y are coherent, then all the tokens in Z are plausible.

A frequently used instance the notion of default is the case when Z is a singleton set $\{a\}$. In this case we omit the set brackets.

The first time that a default rule is applied, the information in X is usually known. However the (uncertain) information Z in the conclusion of a default rule can serve as precondition for further applications of default rules. Therefore, to allow for iterated applications of default rules, the propositions in X are thought of as merely plausible (strict information is always plausible).

Example. We first show how a standard form of Reiter's default logic is one example of our definition. Let A consist of the closed sentences of first-order logic. Let Con be the collection of consistent finite subsets of A in the usual sense, and let $\vdash$ be ordinary logical consequence. Then if $X = \{\alpha\}$, $Y = \{\beta\}$, and $Z = \{\gamma\}$, then the rule $X : Y/Z$ is the rule

$$\frac{\alpha : \beta}{\gamma}$$

where α, β, and γ are sentences. This is the usual form of Reiter's system in virtually all examples. The general format of default rules does allow multiple formulas in the Y position of the defaults. Our system does not deal with this case, since it does not arise in examples. (More on this below.)

Following the usual terminology, we call X the precondition, Y the justification, and Z the conclusion of the default. It is tempting, and in fact allowable, to think of defaults as rules. However, we are officially just dealing with the ternary relation Δ, with no conditions on it. Later, we will define some "transitive" relations generated by Δ and $\vdash$. But we want to consider Δ itself to be rule-like in character, because intuitively it represents empirically given ways of constructing a belief space. So we will speak of elements of Δ as *rules* as well as defaults.

In contrast, the entailment relation $\vdash$ captures common properties of sound informational inferences in the monotonic world. Thus, if we have the information a and b, then certainly we have the information b. One could think of $X \vdash a$ as a default rule $\dfrac{X : \emptyset}{a}$ without justification. It is then reasonable to require

$$X \vdash a \Rightarrow \frac{X : \emptyset}{a} \in \Delta.$$

This requirement, though, will not be necessary for any of our results.

When can we "apply" a default rule? Suppose y is a coherent set of tokens. (This means, by convention, that every finite subset of y is in Con.) We think of the set y as being a current "plausible world" for some experimenter. The experimenter can add a piece of information b into the set y if $X \subseteq y$, and

$$\frac{X : Y}{b} \in \Delta \ \& \ Y \cup y \in Con.$$

In general, the new piece of information could make the resulting set $y \cup \{b\}$ incoherent. This is a possibility which we will rule out later by appropriate restrictions on Δ. But for the purposes of defining our notions of extension and dilation, the restriction is not necessary.

Our notion of default rule application also differs from that of standard default logic. In order to apply a rule

$$\frac{\alpha_1,\ \alpha_2,\ \cdots, \alpha_m : \beta_1,\ \beta_2,\ \cdots,\ \beta_n}{\gamma}$$

to y, we require that $\{\beta_1,\ \beta_2,\ \cdots,\ \beta_n\} \cup y$ be consistent rather than $y \cup \{\beta_i\}$ be consistent for each $1 \leq i \leq n$ (which is weaker, since whenever $\{\beta_1, \ldots, \beta_n\} \cup y \in Con$, then for each i, $\{\beta_i\} \cup y \in Con$). The following example shows that it is strictly stronger:

Example Consider the default information structure

$$(\{a,\ b,\ c\}, \Delta,\ \vdash)$$

where

$$\Delta = \{\frac{: a,\ b}{c}\},$$

and $\vdash$ is generated by (is the smallest entailment relation such that) $\{a,\ b\} \vdash \emptyset$. In this case c can be derived from $\emptyset$ by applying the default rule if we only require a and b be consistent separately. However, c cannot be derived if we insist that $\{a,\ b\}$ be consistent.

Our feeling is that the joint consistency of the set $y \cup \{\beta_1, \ldots, \beta_n\}$ is a more reasonable condition than separate consistency for application of the default rule to the set y. Support for this position comes from the paper of Reiter and Criscuolo [14]. This was the original paper introducing semi-normal default theories. Although the definition of general default rules in that paper refers to separate consistency of each of the justifications, none of the examples ever use it. All of them require joint consistency: that a conjunction of justifications be consistent. In particular, the definition of a semi-normal default rule (Section III) requires joint consistency. We conjecture that one can develop a notion of dilation using separate consistency. However, we have no proof of this, and have not yet investigated this direction.

We next define successively more limited classes of default information structures.

Definition 3.2 *A default rule* $\dfrac{X : Y}{Z}$ *is*

- **rational** *if* $X \cup Y \vdash Z$*;*
- **semi-normal** *if* $Z \subseteq Y$*;*
- **normal** *if* $Y = Z$*;*

- **precondition-free** *if* $Y = Z$ *and* $X = \emptyset$.

A default information structure is rational if all its rules are rational, and similarly for the other cases.

More examples. (1): The eight queens problem. We have in mind in an 8×8 chessboard, so let $8 = \{0, 1, \ldots, 7\}$. Our token set A will be 8×8. A subset X of A will be in Con if it corresponds to an admissible placement of up to 8 queens on the board. For defaults Δ we take the singleton sets

$$\{\frac{: \{\langle i,j\rangle\}}{\{\langle i,j\rangle\}} \mid \langle i,j\rangle \in 8 \times 8\}.$$

We may take $\vdash$ to be trivial: $X \vdash \langle i,j\rangle$ iff $\langle i,j\rangle \in X$.

Example (2): Default approximate reals. Use the information system described above. We might like to say that "by default, a real number is either between 0 and 1, or is the number π". We could express this by letting Δ consist of the rules $\frac{: a}{a}$, where a ranges over rational pairs $\langle p,q\rangle$ such that $p \leq 0$ and $q \geq 1$, together with those pairs $\langle r,s\rangle$ such that $r < \pi$ and $s > \pi$.

4 Extensions

First we recall the notion of "deductive closure", associated with standard information systems[2]. Let $(A, \vdash)$ be a deterministic sequent structure, and G a consistent subset of A. $\overline{G}$, the deductive closure of G, is the set

$$\{a \mid \exists X \subseteq^{fin} G.\ X \vdash a\}.$$

Proposition 4.1 $\overline{G}$ *is an (ideal) element of* $(A, \vdash)$.

Proof Recall that when we say that G is consistent, we mean that for every finite subset X of G, $X \not\vdash \emptyset$. We have to check two things: one is that $\overline{G}$ remains consistent, and the other is that it is deductively closed, i.e., $\overline{\overline{G}} = \overline{G}$.

Let X be a finite subset of $\overline{G}$. Then for each element c of X, there is a finite subset X_c of G such that $X_c \vdash c$ (Take X_c to be $\{c\}$ if $c \in G$). Let $X' = \bigcup_{c \in X} X_c$. By the weakening law for sequent structures we have

$$\forall c \in X.\ X' \vdash c.$$

The second conclusion of Theorem 2.1 asserts that whenever $X \vdash Z$, we also have $X' \vdash Z$. However, X' is a subset of G. Therefore $\overline{G}$ is both consistent and deductively closed.

[2]This terminology is well- entrenched in the domain theory literature. But it is misleading, since we are considering a constructive model-building process, not a process of proof.

$\square$

It is easy to see that the deductive closure operator plays a role analogous to that of the *theory* operator Th in first-order logic. But, as we have warned, this is only an analogy.

We can now define extensions for a default information structure. (From now on, we will use default rules with single tokens as conclusions.)

Definition 4.1 *Let $\underline{A} = (A, \Delta, \vdash)$ be a default information structure, and x an element of $|\underline{A}|$. Define, for a set S of consistent propositions of A, $\Gamma_x(S)$ to be the smallest set such that*

- $x \subseteq \Gamma_x(S)$,
- $\overline{\Gamma_x(S)} = \Gamma_x(S)$,
- $\{a \mid \dfrac{X : Y}{a} \in \Delta \ \&\ X \subseteq \Gamma_x(S) \ \&\ Y \cup S \in Con\} \subseteq \Gamma_x(S)$.

Call E an extension of x if it is a fixed point of the operator Γ_x, i.e. $\Gamma_x(E) = E$.

This definition makes sense when x is any set, consistent or not, but we do not need to consider inconsistent x's. When we write $\Gamma_x(S)$, we will intend that x be consistent. Observe that Γ_x need not be monotonic in S. Observe also that an extension is not necessarily unique.

Examples. (1) Refer to the eight queens problem in the last section. Extensions are those placements of queens which do not violate any constraints of the rules of chess, and which cannot be augmented without causing a violation.

(2). Consider the default reals. In the ideal domain, we refer to elements by the real intervals to which they are isomorphic. The only extension of $[-1, 2]$ would be $[0, 1]$; the interval $[-2, 0.5]$ would have $[0, 0.5]$ as an extension, and there would be 2 extensions of $[-2, 4]$, namely $[0, 1]$ and $[\pi, \pi]$.

These examples are intended to wean the reader away from the view of defaults as default logic. In the eight queens problem, it seems desirable to have a language for reasoning about differing placements. We could of course capture an individual placement $\langle ij \rangle$ of a queen as a proposition: a queen is at row i and column j. But it seems that what we want to do with a logic would be to reason about sets of individual placements. For example, given a placement, is there an extension to a complete placement with no diagonal attacks? The idea of "complete placement" in this example is captured by the original notion of extension due to Reiter. But now the notion is used model-theoretically.

Next we give a definition of another, perhaps more intuitive, operator, used in the first-order case by Reiter to characterize extensions. Reiter shows that the two operators have the same fixed points; we show that they are the same operator.

Definition 4.2 *Let $\underline{A} = (A, \Delta, \vdash)$ be a default information structure, and x a member of $|\underline{A}|$. For any subset S, define $\Phi_x(S)$ to be the union $\bigcup_{i \in \omega} \phi_S^x(i)$, where*

$$\phi_S^x(0) = x,$$
$$\phi_S^x(i+1) = \overline{\phi_S^x(i)} \cup \{a \mid \frac{X : Y}{a} \in \Delta \ \&\ X \subseteq \phi_S^x(i) \ \&\ Y \cup S \in Con\}.$$

It can be easily seen that $\phi^x_S(i)$'s are monotonic in i, a fact that will be used from time to time in the proofs to follow.

Theorem 4.1 *Let $\underline{A} = (A, \Delta, \vdash)$ be a default information structure, and $x \in |\underline{A}|$. Then for every consistent subset S of A,*

$$\Gamma_x(S) = \Phi_x(S).$$

Proof First we show that $\Gamma_x(S) \subseteq \Phi_x(S)$ for every S, given the assumptions stated in the theorem. It is easy to see that $\Phi_x(S)$ is a deductively closed set containing x. Moreover, if $X \subseteq \Phi_x(S)$ and $Y \cup S \in Con$ for some default $\dfrac{X : Y}{a} \in \Delta$, then $X \subseteq \phi^x_S(i)$ for some i (since $\phi^x_S(j)$'s is a monotone sequence in j, and X is finite). That means $a \in \phi^x_S(i+1) \subseteq \Phi_x(S)$. Thus $\Phi_x(S)$ has all the properties mentioned in Definition 4.1. Therefore, $\Gamma_x(S) \subseteq \Phi_x(S)$, since $\Gamma_x(E)$ is the smallest set with these properties.

On the other hand, it follows from the definition of the operator Γ_x that $x \subseteq \Gamma_x(S)$. Therefore, $\phi^x_S(0) \subseteq \Gamma_x(S)$. Assume $\phi^x_S(i) \subseteq \Gamma_x(S)$. Under this assumption, $X \subseteq \phi^x_S(i)$ implies $X \subseteq \Gamma_x(S)$. Recall that

$$\phi^x_S(i+1) = \overline{\phi^x_S(i)} \cup \{a \mid \frac{X : Y}{a} \in \Delta \ \&\ X \subseteq \phi^x_S(i) \ \&\ Y \cup S \in Con\}.$$

Clearly $\overline{\phi^x_S(i)} \subseteq \Gamma_x(S)$, since $\Gamma_x(S)$ is deductively closed. Moreover,

$$\begin{aligned} &\{a \mid \frac{X : Y}{a} \in \Delta \ \&\ X \subseteq \phi^x_S(i) \ \&\ Y \cup S \in Con\} \\ &\quad \subseteq \{a \mid \frac{X : Y}{a} \in \Delta \ \&\ X \subseteq \Gamma_x(S) \ \&\ Y \cup S \in Con\} \\ &\quad \subseteq \Gamma_x(S). \end{aligned}$$

Therefore, $\phi^x_S(i+1) \subseteq \Gamma_x(S)$. By mathematical induction, we have $\bigcup_{i\in\omega} \phi^x_S(i) \subseteq \Gamma_x(S)$, i.e. $\Phi_x(S) \subseteq \Gamma_x(S)$.

□

This theorem, showing that the two operators Γ_x and Φ_x are identical, is much stronger than Reiter's characterization, which merely asserts that they have the same fixed points. What is gained here is a deeper understanding of extensions, and also a conceptually simpler proof of Reiter's result, stated as the following corollary.

Corollary 4.1 (Reiter) *Let $\underline{A} = (A, \Delta, \vdash)$ be a default information structure. A subset E is a fixed point of Γ_x if and only if it is a fixed point of Φ_x. More specifically, E is an extension of x if and only if*

$$E = \bigcup_{i\in\omega} \phi^x_E(i).$$

The following result will be used later.

Theorem 4.2 *With respect to a default information structure* $\underline{A} = (A, \Delta, \vdash)$*, if* y *is an extension of* x*, then* y *is consistent.*

Proof This is because the inconsistency of y implies $y = \overline{x} = x$, since no default rule can ever be applied.

$\square$

5 Dilations

We claimed at the beginning that extensions do not exist, even in reasonable cases. Here is a story to illustrate our point.

> **Story.** There are three couples: Jimmy and Rosalind, Ronnie and Nancy, and George and Barbara. In each couple, the partners are at war. No individual will ever appear at the same party to which the opposing individual was invited. On the other hand, Jimmy has a mild interest in Nancy, Ronnie with Barbara, and George with Rosalind. Thus it is plausible that these people should be seen in company together, and if there is a party to which all three couples are invited, arrangements might be made to go along with a person in whom one has a mild interest, because one should go to a party with a companion.
>
> We are Washington Post reporters, and have heard through the grapevine that a party occurred, to which important people were invited in couples. We are interested in just which males from the above set of people might have attended the party, taking into account that the war conditions definitely hold, and that the mild side interests are plausible, perhaps even likely. Of course we have no real information about the party.

This story leads to a reasonable default information structure with no extensions. Here is how it might be represented. We let j, ros, ron, n, g, and b represent the information that the various individuals attended the party. The war information is represented as follows: $\vdash$ is generated from

$$\{j, ros\} \vdash \emptyset, \{ron, n\} \vdash \emptyset, \{g, b\} \vdash \emptyset.$$

The side interests and their effect on attendance are modeled by the following defaults:

$$\Delta = \{\frac{: j, n}{j}, \frac{: ron, b}{ron}, \frac{: g, ros}{g}\}.$$

This default information structure has no extension for $\emptyset$, the condition of having no information about the party. It turns out, though, that any two-element subset of the males is a dilation of the empty set, as is the set of all the males.

One key to our definition is that dilations, in contrast to extensions, need only be *closed* under the various rules; the other key is that we define the relevant closure operator on consistent subsets of all *plausible consequences* of a given consistent set. Then take any fixed point of the operator to form a dilation.

Definition 5.1 *Let* $\underline{A} = (A,\ \Delta,\ \vdash)$ *be a default information structure, and let* $X \subseteq A$ *be a finite consistent set. We say that* X *directly plausibly entails* b*, written* $X \vdash^1_\Delta b$*, if* $X \vdash b$*, or*

$$\frac{Y : Z}{b} \in \Delta \ \&\ Z \cup X \in Con$$

for some $Y \subseteq X$*. Call* b *a plausible consequence of* X*, written* $X \vdash_\Delta b$ *if*

$$\begin{array}{l} \exists b_1,\ b_2, \cdots, b_n, \quad \textit{such that} \\ X \vdash^1_\Delta b_1, \\ X \cup \{b_1\} \vdash^1_\Delta b_2, \\ \qquad\qquad\qquad \cdots \\ X \cup \{b_1,\ \cdots,\ b_{n-1}\} \vdash^1_\Delta b_n, \\ \ \textit{and} \ \ b = b_n. \end{array}$$

It is important to keep in mind that although plausible consequence looks like a candidate for a non-monotonic entailment relation (as in e.g. [10] or [7]), we are not concerned with the axiomatic aspects of such a relation. It is used here only as part of a convenient notation for our model-building procedure[3].

For any set X, finite or not, we write $T_\Delta(X)$ for the collection of indirect plausible consequences of X:

$$T_\Delta(X) = \{b \mid b \text{ is an indirect plausible consequence of } Y \text{ for some } Y \subseteq^{fin} X\}.$$

Note that $T_\Delta(X)$ represents the collection of all indirect plausible consequences of starting from X. Thus it is usually not consistent. But it allows us to make our main definitions, as follows.

Definition 5.2 *Let* $\underline{A} = (A, \Delta, \vdash)$ *be a default information structure, and* $x \in |\underline{A}|$*. Given any subset* S *of* $T_\Delta(x)$*, define* $\Pi_x(S)$ *to be the set*

$$\overline{S \cup x} \cup \{a \mid \frac{X : Y}{a} \in \Delta \ \&\ X \subseteq S \ \&\ Y \cup S \in Con\}.$$

Definition 5.3 *For a default information structure*

$$\underline{A} = (\ A,\ \Delta,\ \vdash),$$

we say y *is a dilation of* x*, written* $x\ \delta_{\underline{A}}\ y$*, if* x *is consistent and deductively closed:* $\overline{x} = x$*; and* y *is a consistent, fixed point of* Π_x*. We also call* (x, y) *a dilation pair when* y *is a dilation of* x*.*

[3]It is easy to see that the relation of plausible consequence satisfies a Cut law: if $X \vdash_\Delta a$ and $X, a \vdash_\Delta b$ then $X \vdash_\Delta b$. This is essentially just a restatement of the transitivity of plausible consequence. But plausible consequence need not satisfy a Cautious Monotony law: if a and b are both plausible consequences of X, it need not be the case that b is a plausible consequence of X and a. Since our tokens are not thought of as logical formulas, the exact laws they satisfy are not really a concern.

We will often omit the subscripts in the relation $\delta_{\underline{A}}$. It is normal for an information state x to have many dilations y. By the standard notion of a relation, δ stands for the set of dilation pairs associated with a default structure. Note that if y is a fixed point of Π_x, it is automatically deductively closed. Therefore, when $x \; \delta \; y$, both x and y are elements of $\underline{A}$.

The non-monotonic property of default information structures is captured by graphing the relation δ. It is easy to show that the first component can increase while the second component decreases.

Rational Default Structures

One might hope that dilations existed for all default structures. Unfortunately this is not the case in general.

Example Let $\underline{A} = (A, \Delta, \vdash)$ be a default structure where

$$\begin{aligned} &A = \{a, c, e\}, \\ &\Delta = \{\frac{\{a\}:}{c}, \frac{\{a\}:}{e}\}, \quad \text{and} \\ &\vdash \quad \text{is generated by} \quad \{c,\, e\} \vdash \emptyset. \end{aligned}$$

There is no dilation of $\{a\}$. This is because the only candidates would be $\{a, c\}$ and $\{a, e\}$. However, neither of these is a fixed point, let alone a consistent fixed point, of $\Pi_{\{a\}}$. The problem is, of course, that c and e are incompatible, but both are in some sense derivable from a using the defaults. Since the default rules have empty justifications, they should function something like monotonic rules. But if these defaults were part of the $\vdash$ relation, we then would violate the definition of information system.

Our feeling is that the above system *should* have no dilations, and that we should restrict default structures as follows. In a default rule

$$\frac{X : Y}{a},$$

there should be some logical relation among X, Y, and a. It is reasonable to assume that for any default rule of the above form we also have $X, Y \vdash a$. In this case, we call a default information structure *rational*. Intuitively, a rational rule says that if the "justification" Y is not only consistent with our belief, but can be used as part of a monotonic inference of a, then we can also believe a. Insisting that rules be rational is to say that default entailment should agree with monotonic entailment whenever one is applying a default rule locally.

To prove the existence of dilations, we show that semi-normal default rules of the above form with a, the conclusion, in Y, are general enough.

Theorem 5.1 *For every rational default information structure, there is a semi-normal default information structure with exactly the same dilations.*

Proof Let $\underline{A} = (A,\ \Delta,\ \vdash)$ be a rational default information structure.

Consider the semi-normal structure $\underline{B} = (A,\ \Delta',\ \vdash)$, where

$$\Delta' = \{\frac{X : Y \cup \{a\}}{a} \mid \frac{X : Y}{a} \in \Delta\}.$$

Since the only difference between these structures is in the default rules, it is enough to show that $Q \vdash_\Delta a$ in $\underline{A}$ if and only if $Q \vdash_\Delta a$ in $\underline{B}$. Let $X \subseteq Q$, $\frac{X : Y}{a} \in \Delta$, and $Y \cup Q \in Con$. Then $\frac{X : Y \cup \{a\}}{a} \in \Delta'$. Moreover, $X \cup Y \in Con$ and $X \cup Y \vdash a$. Let $X_0 \subseteq^{fin} Q$. It follows from $Y \cup Q \in Con$ and $X \cup Y \vdash a$ that $X \cup X_0 \cup Y \vdash a$ and $X \cup X_0 \cup Y \in Con$. From properties of information systems we deduce that $X \cup X_0 \cup Y \cup \{a\} \in Con$. This implies

$$Y \cup \{a\} \cup Q \in Con$$

because every finite subset of this set is in Con. We have shown that $Q \vdash_\Delta a$ in $\underline{A}$ implies $Q \vdash_\Delta a$ in $\underline{B}$. The other way round is similar but easier.

Using the same argument, one can show that

$$\{a \mid \frac{X : Y}{a} \in \Delta \ \&\ X \subseteq S \ \&\ Y \cup S \in Con\}$$

$$= \{b \mid \frac{X' : Y'}{b} \in \Delta' \ \&\ X' \subseteq S \ \&\ Y' \cup S \in Con\}.$$

Thus $\underline{A}$ and $\underline{B}$ have the same set of dilation pairs.

□

To show existence of dilations in rational structures, it is therefore sufficient to consider semi-normal ones.

Theorem 5.2 *Suppose $\underline{A}$ is a semi-normal default information structure. We have*

$$\forall x \in |\underline{A}|\ \exists y\ x\ \delta\ y.$$

Proof Take y to be a maximal, consistent subset of $T_\Delta(x)$ containing x. Since y is maximal, it is deductively closed. We show that it is also closed under semi-normal default rules. Indeed, let $X \subseteq y$ and $\frac{X : Y}{b} \in \Delta$ such that $Y \cup y \in Con$. Since b is in Y, $y \cup \{b\} \in Con$. Therefore $b \in y$, as y is a maximal consistent subset of $T_\Delta(x)$.

□

Remark. The existence of a maximal consistent subset of $T_\Delta(x)$ follows from Zorn's lemma; the partial order involved is the collection of consistent subsets of $T_\Delta(x)$ under set inclusion. Any chain of this partial order clearly has a maximal element, which is just the union of the chain.

Corollary 5.1 *Dilations always exist for rational default information structures.*

What properties can we show of the dilation relation? For one thing, it is transitive.

Theorem 5.3 *If $x\ \delta\ y$ and $y\ \delta\ z$, then $x\ \delta\ z$.*

Proof. We have to prove that z is a consistent fixed point of Π_x. We first show that $z \subseteq T_\Delta(x)$. Note that $y \subseteq T_\Delta(x)$, because y is a fixed point of Π_x. Therefore, by our observations above, $T_\Delta(y) \subseteq T_\Delta(T_\Delta(x)) = T_\Delta(x)$, and since $z \subseteq T_\Delta(y)$ we have our claim.

Now notice that $z \subseteq \Pi_x(z) \subseteq \Pi_y(z) = z$. This gives the result immediately.

□

From the above proof it can be seen that whenever $x \subseteq y$, $z \subseteq T_\Delta(x)$, and $y\ \delta\ z$, then it follows that $x\ \delta\ z$.

The following theorem gives two more basic properties of the dilation relation.

Theorem 5.4 *For a semi-normal (and therefore for a rational) default information structure $\underline{A}$, we have*

1. $x\ \delta\ y \Longrightarrow y\ \delta\ y$.
2. $\forall x' \in |\underline{A}|\ [(y \supseteq x' \supseteq x\ \&\ x\ \delta\ y) \Longrightarrow \exists y' \supseteq y\ x'\ \delta\ y']$.

Proof (1) This is because y is deductively closed. (2) Since $x \subseteq x'$, $T_\Delta(x) \subseteq T_\Delta(x')$. Clearly y is also a consistent subset of $T_\Delta(x')$. It is now clear that we can take y' to be a maximal consistent subset of $T_\Delta(x')$ extending y.

□

6 Extensions, Dilations, and Modified Extensions

In this section we compare dilations with Reiter-style extensions. We also compare dilations with a modified notion of extension due to Lukaszewicz [9].

6.1 Minimality of Extensions

Given a default information structure $\underline{A} = (A, \Delta, \vdash)$, what exactly is the relationship between dilations and extensions? Theorem 4.1 concludes that an extension for x is just a fixed point of the operator Φ_x. Since Φ_x is defined on the powerset of A, it is also defined on the powerset of $T_\Delta(x)$, a subset of A. A natural question: if y is an extension of x, is y a dilation of x? We will answer this question positively, showing that our notion of dilation is indeed a robust generalization of the notion of extension.

The following proposition is immediate in light of Theorem 4.1, since for every i, $\phi^x_y(i) \subseteq T_\Delta(x)$ when y is an extension of x.

Proposition 6.1 *A subset y is a fixed point of Φ_x if and only if it is one for Φ_x when it is restricted to $T_\Delta(x)$.*

This leads us to the answer to our question above.

Theorem 6.1 *Every extension y of x is also a dilation of x.*

Proof It is enough to show $\Pi_x(y) = y$, provided that $\Phi_x(y) = y$. By Theorem 4.2, y is consistent. It is clearly deductively closed, and it contains x. Suppose

$$\frac{X : Y}{a} \in \Delta, \; X \subseteq y, \quad \text{and } Y \cup y \in \mathit{Con}.$$

Because $\Phi_x(y) = y$, and X is finite, $X \subseteq \phi^x_y(i)$ for some $i \geq 0$. Therefore $a \in \Phi_x(y)$. Thus y is a fixed point of Π_x.

$\square$

More interestingly, every extension of x is a *minimal* fixed point of Π_x. Thus an extension provides a minimal dilation for a given x.

Theorem 6.2 *If y is an extension for x, and (x, z) is a dilation pair such that $z \subseteq y$, then $y = z$.*

Proof We show, by mathematical induction on i, that $\phi^x_y(i) \subseteq \Pi_x(z)$ for every $i \geq 0$. The base case is trivial. Assume $\phi^x_y(i) \subseteq \Pi_x(z)$ for some i. By definition,

$$\phi^x_y(i+1) = \overline{\phi^x_y(i)} \cup \{a \mid \frac{X : Y}{a} \in \Delta \;\&\; X \subseteq \phi^x_y(i) \;\&\; Y \cup y \in \mathit{Con}\}.$$

Clearly $\overline{\phi^x_y(i)} \subseteq \Pi_x(z)$ since $\Pi_x(z)$ $(= z)$ is deductively closed. Furthermore,

$$\frac{X : Y}{a} \in \Delta \;\&\; X \subseteq \phi^x_y(i) \;\&\; Y \cup y \in \mathit{Con}$$

implies

$$\frac{X : Y}{a} \in \Delta \;\&\; X \subseteq \Pi_x(z) \;\&\; Y \cup z \in \mathit{Con}$$

as $z \subseteq y$. Therefore, $\phi^x_y(i+1) \subseteq \Pi_x(z) = z$.

We conclude by mathematical induction that $y \subseteq z$, and by the hypothesis of the theorem that $y = z$.

$\square$

The converse need not be true: a minimal fixed point of Π_x is not necessarily an extension of x. This is because for finite rational default structures a minimal fixed point of Π_x always exists, while Γ_x may not have any fixed point.

The previous theorem implies a property of conventional extensions themselves, which is known for first order default theories from Reiter's work.

Corollary 6.1 *If y_1 and y_2 are both extensions of x such that $y_1 \subseteq y_2$, then $y_1 = y_2$.*

The following theorem indicates that extensions have a rather strong transitivity property.

Theorem 6.3 *If y is an extension of x and z is an extension of y, then $y = z$.*

The proof of this theorem is not hard. It can be done by mathematical induction as in the proof of Theorem 6.2. However, it is a bit of a surprise to us that we could not find the result in the literature.

6.2 Modified Extensions

We next compare our notion of dilation with the concept of modified extensions in the work of Lukaszewicz. This account can be found in Besnard's book on default logic [1]. (Two very much related concepts are the idea of J-extensions from the work of Delgrande and Jackson [5], and the idea of assertional default theories due to Brewka [4].) Etherington [6] further elaborates on extension existence problems. The idea in the work of Lukaszewicz is that the application of a default rule fails, when the consequent of that rule would contradict the justification of a default rule that had already been applied.

The way that Lukaszewicz's system works can be illustrated by our Washington Post story above. Recall that $\vdash$ is generated from

$$\{j, ros\} \vdash \emptyset, \{ron, n\} \vdash \emptyset, \{g, b\} \vdash \emptyset.$$

We also have the following defaults:

$$\Delta = \{\frac{: j, n}{j}, \frac{: ron, b}{ron}, \frac{: g, ros}{g}\}.$$

Now, if we start with the empty set, the first default is applicable, and we get j as part of a putative extension. However, we could not then, in Lukaszewicz's system, apply the second default, because the conclusion of that default (ron) is incompatible with the justification n in the first default. Notice that we also can not apply the third default, because its justification is already incompatible with an already established conclusion. So, if we choose the first default to apply, we get a modified extension $\{j\}$. Similarly, if we were to choose one of the other defaults to apply first, we would get the modified extensions $\{ron\}$ and $\{g\}$.

We omit the original definition of modified extension, and instead present the construction analogous to Reiter's Φ_x operator. It is a function on pairs of token sets to pairs of token sets. The first set in the pair keeps track of consequents, and the second keeps track of justifications.

Definition 6.1 *Let S and T be sets of tokens of a default information structure, and fix an ideal element x. Define the operator $M_x(S, T)$ inductively as follows.*

- *Let $S_0 = x$ and $T_0 = \emptyset$;*

- *Let*

$$S_{i+1} = \overline{S_i} \cup \{a \mid \frac{X : Y}{a} \in \Delta, X \subseteq S_i, \ \textit{and}\ (S \cup a \cup T \cup Y) \in Con\};$$

- *Let*

$$T_{i+1} = T_i \cup \{b \mid \frac{X : Y}{a} \in \Delta, X \subseteq S_i, b \in Y, \ \textit{and}\ (S \cup a \cup T \cup Y) \in Con\}.$$

Then put

$$M_x(S,T) = \left(\bigcup_{i=1}^{\infty} S_i, \bigcup_{i=1}^{\infty} T_i\right).$$

Finally, E is a modified extension of x if for some T,

$$M_x(E,T) = (E,T).$$

Example. Referring to the Washington Post story,

$$M_\emptyset(\{j\},\{n\}) = (\{j\},\{n\}).$$

Is there any relation between dilations and modified extensions? Lukaszewicz has shown that modified extensions exist for *any* default theory. It is not hard to show that the proof goes through for any default information system, even a nonrational one. So in this respect, modified extensions have an advantage over dilations. The advantage of dilations over extensions and modified extensions is rather in the locality of computation: in calculation the iterations for extensions and modified extensions, one continually has to check consistency with the final answer. In contrast, each iteration in calculating the set of plausible consequences can be generated purely on the previous iterations. The fixed point need only be taken at the end.

Consider another example: let $\{a,b\} \vdash \emptyset$, and let

$$\Delta = \{\frac{: a}{b}\}.$$

Then $\{b\}$ is the unique dilation of $\emptyset$, while $\emptyset$ is the unique modified extension of $\emptyset$. This shows that modified extensions are "conservative", while dilations are "brave". It also shows that there need not be any inclusion relationships between modified extensions and dilations.

Besnard gives an example of a default theory with no maximal modified extension. In contrast, every ideal element will always have a maximal dilation in a rational default system, as our existence proof shows.

We feel that choosing between modified extensions and dilations will be a matter of taste. The real point is not that one system has priority over the other, but that both systems can be considered model-theoretically. To illustrate the point, let us consider another example from Lukaszewicz, as quoted in Besnard [1].

Example Normally on Sundays I go fishing, unless I sleep late. Also, normally on holidays, I sleep late unless I go fishing. We represent these sentences by the following pair of semi-normal default rules:

$$\frac{s : f, w}{f}, \frac{h : \neg w, \neg f}{\neg w}$$

where w stands for "waking early", and the negated letters are just new tokens which clash with their positive versions. Now suppose our starting state of information is

$$x = \{s, h\}.$$

There are then two extensions of x: $\{s, h, f\}$, and $\{s, h, \neg w\}$. If we think of these extensions as theories, then the models of both theories (the ones which support the most reliable conclusions) are those which satisfy all of s, h, f, and $\neg w$. Such models violate the justifications of both defaults – a fact which led to the definition of modified extensions.

What if we think of these two extensions as themselves being models? Admittedly, they are partial models, as they may not settle facts not about fishing and sleeping situations. But if we ask what formulas are supported in both models, we get pleasing results. For example, the formula $f \vee \neg w$ is supported in both models. Intuitively this says that if it is both a Sunday and a holiday, then I will either sleep late or go fishing. This interpretation seems to us to be exactly right.

6.3 Dilations in precondition-free default structures

Recall that a default information structure is *precondition-free* if each default rule in Δ is of the form

$$\frac{: a}{a}.$$

Such default rules are interesting because they are relatively expressive, and, moreover, extensions always exist.

Theorem 6.4 *Let* $(A, \Delta, \vdash)$ *be a precondition-free default information structure. We have*

$$x \; \delta \; y \iff y \textit{ is an extension of } x.$$

Proof Theorem 6.1 shows that every extension determines a dilation. Therefore, it is enough to show that whenever $x \; \delta \; y$, y is an extension of x. For any dilation pair (x, y), we have $y = \Pi_x(y)$, i.e.

$$y = \overline{y \cup x} \cup \{a \mid \frac{: a}{a} \in \Delta \;\&\; y \cup \{a\} \in Con\}.$$

By observing that $\Phi_x(y) = \phi_y^x(0) \cup \phi_y^x(1) \cup \phi_y^x(2)$, where

$$\phi_y^x(0) = x,$$
$$\phi_y^x(i+1) = \overline{\phi_y^x(i)} \cup \{b \mid \frac{: b}{b} \in \Delta \;\&\; y \cup \{b\} \in Con\},$$

one can easily show that $y = \Phi_x(y)$. Thus y is an extension of x.

$\square$

This result cannot be generalized to normal default information structures, ones for which the precondition is allowed to be some finite set X, but where the justification and conclusion are the same singleton set a. For such structures, dilations and extensions can still be different.

Example Let $(A,\ \Delta,\ \vdash)$ be a default information structure, where

$$A = \{a_1,\ a_2,\ b\},$$
$$\Delta = \{\frac{:a_1}{a_1},\ \frac{\{a_1\} : a_2}{a_2},\ \frac{:b}{b}\},$$
$$\text{and } \vdash \text{ is generated by } \{a_1, b\} \vdash \emptyset.$$

For this default information structure, $\{a_2,\ b\}$ is a dilation of the empty set, but $\{a_2,\ b\}$ is not an extension of $\emptyset$.

7 The Dynamics of Default Information

A *default partial order* is one determined by the componentwise ordering on dilation pairs. We establish various properties of default partial orders and provide some preliminary comparisons with domain theory.

Let $(D, \sqsubseteq)$ be a partial order. A non-empty subset $S \subseteq D$ is called *upper directed* if for every $x, y \in S$, there is a $z \in S$ such that $x \sqsubseteq z$ and $y \sqsubseteq z$. Similarly, one has the dual notion of a lower directed set. D is called upper complete (lower complete) if least upper bound (greatest lower bound) exists for all upper (lower) directed sets.

Theorem 7.1 *For any default information structure $\underline{A}$, the relation δ ordered under componentwise inclusion is an upper complete partial order.*

Proof Since the elements of δ are set pairs and the order is based on set inclusion, we clearly have a partial order.

Now let $\{(x_i, y_i) \mid i \in I\}$ be an upper directed set of δ. We show that $\Pi_x(\bigcup_{i\in I} y_i) = \bigcup_{i\in I} y_i$, (where $x = \bigcup_{i\in I} x_i$) which implies $(\bigcup_{i\in I} x_i,\ \bigcup_{i\in I} y_i)$ is the least upper bound of the upper directed set.

It follows by definition that $\bigcup_{i\in I} y_i \subseteq \Pi_x(\bigcup_{i\in I} y_i)$. On the other hand, suppose $\dfrac{X : Y}{a} \in \Delta$, $X \subseteq \bigcup_{i\in I} y_i$, and $Y \cup \bigcup_{i\in I} y_i \in \mathit{Con}$. Since X, Y are finite and y_i's are upper directed, we have $X \subseteq y_k$ and $Y \cup y_k \in \mathit{Con}$, for some $k \in I$. Therefore $a \in y_k$, which implies $a \in \bigcup_{i\in I} y_i$. From this it is easy to see that, indeed, $\Pi_x(\bigcup_{i\in I} y_i) \subseteq \bigcup_{i\in I} y_i$.

$\square$

However, $(\delta_{\underline{A}}, \sqsubseteq)$ need not be lower complete. Consider the following default information structure.

Example Let $(A,\ \Delta,\ \vdash)$ be a default information structure where

$$
\begin{aligned}
&A = \{(0,\ i) \mid i \in \omega\} \cup \{(1,\ i) \mid i \in \omega\},\\
&\Delta = \{\frac{:(0,\ i)}{(0,\ i)} \mid i \in \omega\} \cup \{\frac{(0,\ i) : (1,\ i)}{(1,\ i)} \mid i \in \omega\},\\
&\text{and } \vdash \text{ is generated from } \{(0,\ i), (1,\ j)\} \vdash \emptyset \text{ whenever } i \neq j.
\end{aligned}
$$

Take $y_i = \{(1,\ j) \mid j \geq i\}$. We have a decreasing chain $(\emptyset,\ y_i)$ of dilation pairs. However, $\bigcap_{i\in\omega} y_i = \emptyset$ and $\emptyset$ is *not* a dilation of $\emptyset$. Therefore, the decreasing chain under consideration does not have a greatest lower bound.

The above example also illustrates that $(\delta_{\underline{A}}, \sqsubseteq)$ need not have a bottom element. In fact, much more complicated behavior is possible – the minimal elements of the partial order can be uncountable, even for countable A, and non-RE, even for recursive A. This can be seen from the following example.

Example Let $(A,\ \Delta,\ \vdash)$ be a default information structure where

$$
\begin{aligned}
&A = \{(0,\ i) \mid i \in \omega\} \cup \{(1,\ i) \mid i \in \omega\},\\
&\Delta = \{\frac{:(0,\ i)}{(0,\ i)} \mid i \in \omega\} \cup \{\frac{:(1,\ i)}{(1,\ i)} \mid i \in \omega\},\\
&\text{and } \vdash \text{ is generated from } \{(0,\ i), (1,\ i)\} \vdash \emptyset \text{ for every } i \in \omega.
\end{aligned}
$$

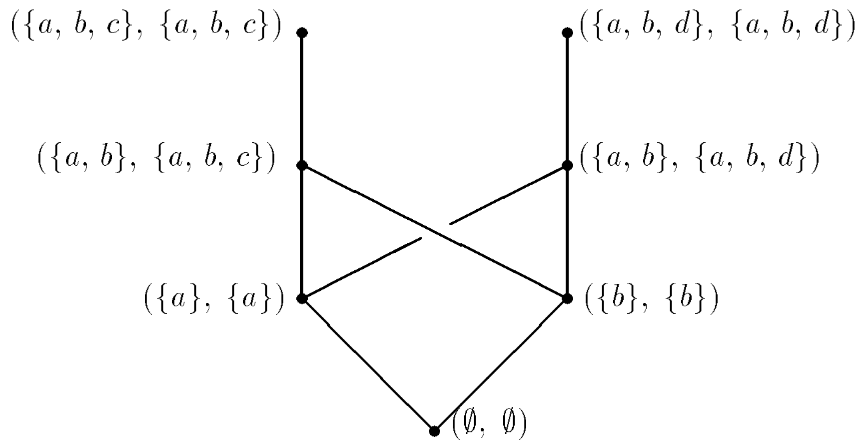
Then for each function $f : \omega \to \{0,\ 1\}$, $(\emptyset,\ \{(f(i),\ i) \mid i \in \omega\})$ is a minimal dilation.

We end this section with a default information structure whose δ represents a finite non-Scott domain.

Example Let $\underline{A} = (A,\ \Delta, \vdash)$ be a default information structure where

$$
\begin{aligned}
&A = \{a,\ b,\ c,\ d\},\\
&\Delta = \{\frac{\{a,\ b\} : c}{c}, \frac{\{a,\ b\} : d}{d}\}, \text{ and}\\
&\vdash \text{ is generated by}\\
&\ \{c\} \vdash a,\ \{c\} \vdash b,\\
&\ \{d\} \vdash a,\ \{d\} \vdash b,\\
&\ \{c,\ d\} \vdash \emptyset.
\end{aligned}
$$

The cpo of dilation pairs can be pictured as follows. It is not a Scott domain.



8 Concluding remarks

The main technical result in the paper is the definition of a notion of "generalized extension" for non-normal default structures, for which we can prove an existence result, and which are computed "locally", in the sense that pieces of information are added to a belief space without having to know what the final belief space is. We also showed the relationship of our notion to the standard notion of extensions: when extensions exist, they are minimal dilations. We also gave a brief comparison to the Lukaszewicz definition of modified extension. We do not want to claim that our notion of dilation is the "right" definition for a belief space, though. Perhaps an even more robust definition can be given. In particular, it might be possible to give a locally computed version of Lukaszewicz's modified extensions.

The real contribution of the paper, we feel, is that we view these fixed point constructions as defining models, not theories. We are taking the first step towards a *partial model theory* for default reasoning. Although default logic itself forms an example of a default information system, we caution the reader not to take this example as a canonical one. Instead, one should think along the lines of denotational semantics for programming languages. There, a program will define, as its semantics, an element of a particular domain (the space of continuous functions from inputs to outputs). In analogy, we view a default information system as generating a relation on a domain: the dilation relation, in general, or the extension relation, in special cases. Just as program specifications are given by a logic of such a domain, we think that a logic for default reasoning should really speak about things like the dilation relation generated by a default information system. We have some results in this direction, which will appear in subsequent papers [17, 16].

Given this new perspective, it is perhaps not absolutely crucial to find "the right" notion of generalized extension, which will form "the" class of belief spaces generated by, say, semi-normal information systems. Our perspective, instead, leads us to other criteria. If default systems are like programs, and the meaning of a program is the function it computes, then the values of the function are finitely approximable as the limit of a sequence of finite partial functions. In the definition of dilation, we have tried to capture this intuition by insisting that every token occurring in a dilation must appear by virtue of a finite chain of default rule applications. Each such rule application is a reflection of a reasoner's experience or preferences, encoded in the default rules of the system. The computations, in this respect, are entirely local; fixed points are taken just once.

In a final belief space (the result of a fixed point construction) we see no reason why intermediate local pieces of information leading to the inclusion of a token in a space need themselves to be consistent with the final space. The intermediate values used in the computation of a function need not ever appear as final outputs; they need only be computed according to locally correct rules. So considerations like those in Delgrande and Jackson [5], and Lukascewicz [9], which are concerned with inclusion of intermediate conclusions and/or justifications in extensions, are for us a bit beside the point. Halpern, in a private communication, has also raised similar issues with us. He has asked why, in fact, one could not take *any* theory consistent with a base theory as an extension. We are inclined to say that Halpern might well be correct, except that the finite approximability property would be lost. Of course, the real point for us is that whatever fixed point definitions we settle on, we need to think of such fixed points as models, not theories.

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