A Note on Maximal Stable Functions

Guo-Qiang Zhang
Department of Computer Science
University of Georgia
Athens, Georgia 30602
U.S.A.

ABSTRACT

This note studies maximal elements in dI-domains, especially maximal stable functions. First, a sufficient condition for a stable function to be maximal is provided. The condition requires that the function be total, in the sense that it preserves maximal elements. Examples are then given to demonstrate that being total is not necessary for a stable function to be maximal. Secondly, a characterization is given for dI-domains for which 'maximal' also implies 'total'. Finally, topological properties of maximal elements are investigated. dI-domains for which maximal is the same as total are those having a certain Hausdorff property.

1 Introduction

In [7] and [9], Girard pointed out that in the category of coherent spaces maximal and total are different notions in general because they have a different logical complexity. He proposed a basic definition for totality:

> If R is a totality candidate for A and S for B then we write $R \to S$ for the set of objects (totality candidates) f of type $A \to B$ such that $a \in R \Rightarrow fa \in S$.

Recently, there has been some interesting work on totality, including the notion of totality using games [1], totality spaces [10], topological notions of totality [3], and the relationship between maximality and termination [5]. However, the connection between totality and maximality remains illusive. This note tries to understand the issue from a domain-theoretic point of view.

We realize that, when studying maximal elements, it is better to use the stable order introduced by Berry [4]. Part of the reason is that if one takes the standard pointwise order on Scott-continuous functions, even the most well-behaved identities are not necessarily maximal. The identities are maximal, however, with respect to the stable order (this fact is also noticed by G.-Y Girard, Y. Lafont, P. Taylor [9], and A. Pitts, as mentioned in [6]).

This note studies maximal elements in the category of dI-domains and stable functions. We give a sufficient condition for a stable function to be maximal. The condition requires that the function be total, in the sense that it preserves maximal elements. Examples are then given to demonstrate that being total is not necessary for a stable function to be maximal. We then provide a characterization of stable domains for which 'maximal' also implies 'total'. Finally, we recapture maximal elements in a topological setting.

2 Traces

We briefly review a generalization of the notion of traces from coherent spaces to dI-domains. It is a necessary step here because the notion of traces is our main technical tool for studying maximal stable functions. For convenience, we use p, q for complete primes; a, b for isolated elements; and D^0, D^p for the collections of isolated elements and complete primes of a dI-domain D, respectively. The stable function space from D to E is written as $[D \to_s E]$. Readers are referred to [4, 11] for dI-domains and the stable order, and [2, 13] for basic domain-theoretic concepts used in this paper.

Definition 2.1 *([11]) Let $f : D \to E$ be a stable function, where D, E are dI-domains. Define μf to be a set of pairs $(a, p) \in D^0 \times E^p$ such that if*

$$f(a) \sqsupseteq p \;\&\; [\forall a' \sqsubseteq a.\ f(a') \sqsupseteq p \Rightarrow a = a'].$$

We call the set μf the trace of f. Stable functions have a property that their values are totally determined by those at some minimal points. One can then understand a pair $(a,\, p) \in \mu f$ as saying a is a minimal point for f to assume value p.

Theorem 2.1 *([11]) Let $f,\, g \in [D \rightarrow_s E]$. We have*

1. *$f \sqsubseteq_s g$ if and only if $\mu f \subseteq \mu g$;*
2. *For any $x \in D$, $f(x) = \bigsqcup \{\, p \mid \exists a \sqsubseteq x.\, (a,\ p) \in \mu f \,\}$;*
3. *A set $\{\, (a_i,\, p_i) \mid i \in I \,\} \subseteq D^0 \times E^p$ is the trace of some $f \in [D \rightarrow_s E]$ if and only if it is*

$$\begin{array}{l} \text{Compatible}: \ \forall J \subseteq^{fin} I.\, \{\, a_i \mid i \in J \,\}\uparrow \Rightarrow \{\, p_i \mid i \in J \,\}\uparrow, \\ \text{Minimal}: \ a_i \uparrow a_j \ \&\ (p_i = p_j) \ \Rightarrow \ (a_i = a_j), \text{ and} \\ \text{Complete}: \ \forall p \in E^p.\, p_i \sqsupseteq p \ \Rightarrow \ \exists j.\, p_j = p \ \&\ a_i \sqsupseteq a_j. \end{array}$$

Coherent spaces form a full subcategory of dI-domains. So all the results in this note will have obvious corollaries for coherent spaces which will therefore not be stated.

3 Maximal Stable Functions

Let D be a dI-domain. An element $m \in D$ is called *maximal* if there is no element in D bigger than it. In flat domains, everything is maximal except the bottom. Write $\mathsf{M}(D)$ for the set of maximal elements of D. Clearly different maximal elements are incompatible. If an element a has the property that any complete prime compatible with it is dominated by it, then a must be a maximal element, and vice versa.

Theorem 3.1 *Let D, E be dI-domains. A stable function f is maximal in $[D \rightarrow_s E]$ if it is total, i.e., for any $m \in \mathsf{M}(D)$, we have $f(m) \in \mathsf{M}(E)$.*

It should be pointed out that Theorem 2.1 realizes maximal stable functions as those whose trace are maximal with respect to set inclusion.

Proof Suppose a stable function $f : D \to E$ maps $\mathsf{M}(D)$ to $\mathsf{M}(E)$. Let g be a stable function such that $f \sqsubseteq_s g$, that is, $\mu f \subseteq \mu g$ by Theorem 2.1. It suffices to show that $\mu f = \mu g$. Assume $(a, p) \in \mu g$, and choose $m \in \mathsf{M}(D)$ with $a \sqsubseteq m$. By Theorem 2.1,

$$f(m) = \bigsqcup\{ p' \mid (a', p') \in \mu f \ \&\ a' \sqsubseteq m \}.$$

For each $(a', p') \in \mu f$ with $a' \sqsubseteq m$, we have $(a', p') \in \mu g$. Hence by the compatibility of μg, the family $(\{p\} \cup X)$ is compatible for any finite subset

$$X \subseteq \{ p' \mid (a', p') \in \mu f \ \&\ a' \sqsubseteq m \}.$$

Consequently,

$$p \uparrow \bigsqcup\{ p' \mid (a', p') \in \mu f \ \&\ a' \sqsubseteq m \},$$

that is, $p \uparrow f(m)$. However, $f(m)$ is maximal by assumption, so

$$p \sqsubseteq \bigsqcup\{ p' \mid (a', p') \in \mu f \ \&\ a' \sqsubseteq m \}.$$

Since p is a complete prime, $p \sqsubseteq p'$ for some p', with $(a', p') \in \mu f$ and $a' \sqsubseteq m$. This means, by the completeness of μf, that $(b', p) \in \mu f$ for some $b' \sqsubseteq a' \sqsubseteq m$, which is only possible when $b' = a$ because $(b', p) \in \mu g$ and μg is minimal (c.f. Theorem 2.1). Therefore $(a, p) \in \mu f$ and $\mu g \subseteq \mu f$.

$\square$

Now we can see that identities are maximal since they are total. In fact many other familiar functions arising from the category of dI-domains are maximal, by their totality property. We have the following corollary.

Corollary 3.1

1. *Identities* $\mathit{id}_D : D \to D$ *are maximal.*
2. *The composition of total functions is maximal.*
3. *Injections* $\mathit{in}_i : D_i \to (D_0 + D_1)$ $(i = 0, 1)$ *are maximal.*
4. *Projections* $\pi_i : D_0 \times D_1 \to D_i$ $(i = 0, 1)$ *are maximal.*

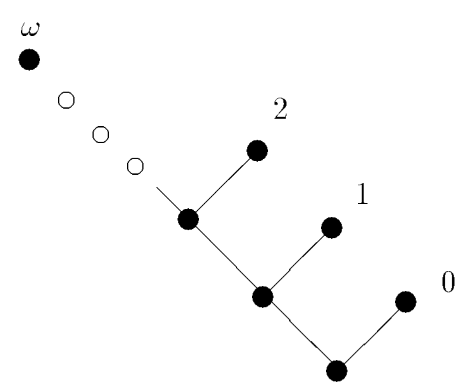
The function *curry* : $(D \times E \to F) \to (D \to [E \to F])$ is maximal for coherent spaces D, E, F. It sends the trace $\{((a,\, b),\, p) \mid ((a,\, b),\, p) \in \mu f\}$ of a function $f: \; D \times E \to F$ to the trace $\{(a,\, \{\, (b,\, p)\,\}) \mid ((a,\, b),\, p) \in \mu f\}$. Therefore, it is total since the coherence relation on tokens of the form $((a,\, b),\, p)$ is the same as the one on those of the form $(a,\, \{\, (b,\, p)\,\})$.

At the moment our technique cannot be applied to the function *app* : $(D \to E) \times D \to E$ directly (however *app* is maximal for coherent spaces D, E, F. Observe that its trace is $\{((\{\, (a,\, p)\,\},\, a),\, p) \mid a \in D^0,\, p \in E^p\})$. The difficulty is that we do not know if $f(a)$ is maximal or not, assuming f and a are. In the following sections we will develop techniques so that we can deal with functions like *app* in general.

4 Example

Totality is not a necessary condition for a stable function to be maximal. We have the following example.

Example: The lazy natural numbers (see the picture below). Let **2** stand for the two-point lattice. Then the function f which maps all natural numbers $0, 1, 2 \cdots$ to the top of **2**, and the rest to the bottom, is clearly maximal with respect to traces. However, f maps the maximal element ω to the bottom.



One hopes this troublesome domain was an isolated incident and it would not arise provided that only 'normal' constructions on 'normal' domains are used. Not so. The following is a list of domains that can cause similar trouble:

- $[\omega_\perp \to_s \mathcal{T}]$, where $\mathcal{T}$ is the truth-value cpo;
- $[\omega_\perp \to_s \omega_\perp]$;
- $[(\omega_\perp \to_s \mathbf{2}) \to_s (\omega_\perp \to_s \mathbf{2})]$; and
- $[\mathbf{2} \times \mathbf{2} \to_s \mathcal{P}(\omega)]$, where $\mathcal{P}(\omega)$ is the cpo of subsets of natural numbers ordered by inclusion.

The next section explains why they can cause trouble.

5 Maximum-separable Domains

Motivated from the example of lazy natural numbers we arrive at the following definition. Our proof reveals the true logical nature of "maximum-separable domains".

Definition 5.1 *A dI-domain D is maximum-separable if*

$$\forall m \in \mathsf{M}(D)\ \exists a \in D^0.\ \uparrow(a) \cap \mathsf{M}(D) = \{m\}.$$

Informally, we call such an a a *separation point* for m. These a's can be regarded intuitively as "name-tags" or "finite witness" for maximal elements. If a is a separation point for $m \in \mathsf{M}(D)$, then for any element $b \in D$, either $a \not\uparrow b$, or $b \sqsubseteq m$. One can readily see that maximum-separable dI-domains have a countable number of maximal elements. The upside down complete binary tree is, therefore, not maximum-separable. The example given in the previous section is not maximum-separable because there are no a's with the required property for ω.

Theorem 5.1 *Fix a dI-domain D. Then D is maximum-separable if and only if for any dI-domain E, we have*

$$f \in \mathsf{M}([D \to_s E]) \Rightarrow \forall m \in \mathsf{M}(D). f(m) \in \mathsf{M}(E).$$

This theorem implies that totality and maximality are equivalent for maximum-separable domains.

Proof *Only if* : Suppose D is maximum-separable and $f \in \mathsf{M}([D \to_s E])$. We show that for any maximal element m in D, $f(m)$ is maximal in E. We assume that the bottom in E is not maximal. Given any $m \in \mathsf{M}(D)$, there must be a separation point a for m such that $(a, p) \in \mu f$. Otherwise, one could carefully augment μf with (a, p) for some p and still get a trace back, which would contradict the maximality of f. Now let q be any complete prime compatible with $f(m)$. It can be seen from Theorem 2.1 that $\{(a, q)\} \cup \mu f$ remains a trace for some stable function, using the fact that some (a, p) is already in μf. The maximality of μf implies that $(a, q) \in \mu f$. Therefore, any complete prime compatible with $f(m)$ must actually be below $f(m)$. That means $f(m) \in \mathsf{M}(E)$.

If : Let m be maximal in D and consider the set $\mathcal{P}$, whose members are families $\{\uparrow(a_i) \mid i \in I\}$ of principal filters in D such that $\uparrow(a_i) \cap \uparrow(a_j) = \emptyset$ if $i \neq j$; *and* $m \notin \uparrow(a_i)$ for all $i \in I$. If this set is empty, then m is the greatest element of D, so D is maximum-separable. Otherwise, the algebraicity of D guarantees that $m \notin \uparrow(a)$ for some $a \in D^0$. Hence $\{\uparrow(a)\} \in \mathcal{P}$. The poset $\mathcal{P}$ is also *inductive*, for the union of a directed family in $\mathcal{P}$ is again a member in $\mathcal{P}$. Using the Axiom of Choice in form of Zorn's Lemma, we obtain a maximal family $\mathcal{F} = \{\uparrow(b_j) \mid j \in J\}$ in $\mathcal{P}$. Then clearly $\{(b_j, \top) \mid j \in J\}$ is the trace of a unique stable function $f \in [D \to_s \mathbf{2}]$. Now, we can choose a maximal function g above f in $[D \to_s \mathbf{2}]$.

By assumption, we know that $g(m) = \top$. Therefore, there exists some finite $a \sqsubseteq m$ such that $(a, \top)$ is in the trace of g. As $b_j \not\sqsubseteq m$ for all $j \in J$, we conclude that a is incompatible with all b_j, for f is below g in the stable order.

If D is not maximum-separable, then we can find some m' maximal in D with $a \sqsubseteq m' \neq m$. Since $\uparrow(a)$ is Scott-open and D is algebraic, we can find some finite b in D such that $a \sqsubseteq b$ and $m \notin \uparrow(b)$ as $m' \not\sqsubseteq m$. Clearly, $\uparrow(b) \cap \uparrow(b_j) = \emptyset$ for all $j \in J$, so the family $\mathcal{F} \cup \{\uparrow(b)\}$ is greater than $\mathcal{F}$, which is a contradiction.

$\square$

As finite dI-domains and lattice dI-domains (dI-domains which are also lattices) are maximum-separable, the following corollaries are immediate.

Corollary 5.1

1. *Let D be a finite dI-domain. For any dI-domain E,*

$$f \in \mathsf{M}([D \to_s E]) \Leftrightarrow \forall m \in \mathsf{M}(D).f(m) \in \mathsf{M}(E);$$

2. *Let D be a lattice dI-domain. Then for any dI-domain E,*

$$f \in \mathsf{M}([D \to_s E]) \Leftrightarrow \forall m \in \mathsf{M}(D).f(m) \in \mathsf{M}(E);$$

3. *A function $f : \omega_\perp \to_s \omega_\perp$ is maximal if and only if it is total;*

4. *Suppose $f \in \mathsf{M}([D \to_s E])$. If $m \in D^0 \cap \mathsf{M}(D)$, then $f(m) \in \mathsf{M}(E)$.*

Part 4 of the last corollary says that maximal stable functions always map *finite* maximal elements to maximal elements. The following proposition is easy to justify.

Proposition 5.1 *Let D, E be maximum-separable dI-domains. Then $D \times E$, $D + E$ and $D_\perp$ are maximum-separable.*

This proposition concludes that except function space, most other familiar constructions preserve maximum-separability. For product, it is actually the case that $\mathsf{M}(D \times E) \cong \mathsf{M}(D) \times \mathsf{M}(E)$, without assuming maximum-separability of D or E.

By the definition of maximum-separability, the dI-domains $[\omega_\perp \to_s \mathcal{T}]$, $[\omega_\perp \to_s \omega_\perp]$, $[(\omega_\perp \to_s \mathbf{2}) \to_s (\omega_\perp \to_s \mathbf{2})]$, and $[\mathbf{2} \times \mathbf{2} \to_s \mathcal{P}(\omega)]$ are not maximum-separable. This means if we take these domains as source of a stable function space, there is some maximal element in the function space which is not total. Note however, the maximal elements are total in these non-maximum-separable domains themselves because $\omega_\perp$, $[\omega_\perp \to_s \mathbf{2}]$, $\mathbf{2} \times \mathbf{2}$ are all maximum-separable.

6 Topological Considerations

As shown in [12], there is no truly topological characterization of stable functions. Although stable neighborhoods, as introduced in [12], have the property that a function is stable if and only if stable neighborhoods are preserved under the inverse image of the function, they do not give rise to a topological space. This is because stable neighborhoods are not closed under arbitrary unions. Nevertheless, they are closed under disjoint unions.

For the same reason, there is no *truly topological* characterization of maximal stable functions. But it would still be informative, or even illuminating to consider the maximality issue in the topological setting. For that purpose, we introduce the notion of a *conjunctive* open set.

Definition 6.1 *A non-empty Scott open set U of D is called conjunctive if the intersection of any two non-empty open subsets of U is always non-empty. We write $\mathcal{C}(D)$ for the collection of conjunctive Scott open sets of D.*

Clearly, conjunctive sets are closed under finite intersection provided that the intersection is non-empty. It is easy to see that any non-empty, open subset of a conjunctive set is still conjunctive. It can also be seen that an open set is conjunctive if and only if it contains a unique maximal element. Therefore, we can also say that an open set is conjunctive if and only if it is directed.

We can recapture maximum-separable domains in terms of conjunctive open sets.

Theorem 6.1 *A dI-domain D is maximum-separable if and only if its maximal elements have the Hausdorff property with respect to $\mathcal{C}(D)$, i.e.,*

$$\forall m_1 \neq m_2 \in \mathsf{M}(D)\ \exists U_1, U_2 \in \mathcal{C}(D).\ [m_1 \in U_1\ \&\ m_2 \in U_2\ \&\ U_1 \cap U_2 = \emptyset].$$

The proof is straightforward; hence, it is omitted.

It is interesting to note that maximal elements can also be characterized by conjunctive open sets. The following result says that an element is maximal exactly when it is the least upper bound of a conjunctive open set.

Theorem 6.2 *Let D be a maximum-separable dI-domain. Then $m \in \mathsf{M}(D)$ if and only if*

$$\exists U \in \mathcal{C}(D).\ m = \bigsqcup U.$$

We omit the easy proof and proceed to consider how maximal stable functions may be determined by topological means. For that purpose we introduce the notion of a *non-degenerating subset* relation on stable neighborhoods. Recall that a stable neighborhood is just a Scott open set generated by a set of pairwise incompatible isolated elements. Therefore, any stable neighborhood U can be expressed as $\uparrow(\mu P)$, the upper closure of the set μP of minimal points of U (which are pairwise incompatible).

Definition 6.2 *For stable neighborhoods U and V, we call U a non-degenerating subset of V if*

$$U \subseteq V \quad \textit{and} \quad \downarrow(U) = \downarrow(V).$$

When U is a non-degenerating subset of V, we write $U \prec V$.

Although U can be smaller when it is a non-degenerating subset of V, both U and V must contain the same maximal elements.

Theorem 6.3 *Given maximum-separable dI-domains D and E, a stable function $f : D \to E$ is maximal if and only if for every pair of stable neighborhoods U, V of E, we have*

$$U \prec V \Rightarrow f^{-1}(U) \prec f^{-1}(V).$$

Of course, Theorem 2.7 in [12] is implicitly used here to ensure that both $f^{-1}(U)$ and $f^{-1}(V)$ are stable neighborhoods.

Proof. 'Only if': Easy.

'If': Suppose $f : D \to E$ has the property that

$$U \prec V \Rightarrow f^{-1}(U) \prec f^{-1}(V)$$

for any stable neighborhoods U, V of E. By Theorem 3.1, it is enough to show that f is total. Let $m \in \mathsf{M}(D)$. Clearly $m \in f^{-1}(\uparrow(\perp_E))$. By Theorem 6.1, there is a stable neighborhood Q_0 with $\mathsf{M}(E) \subseteq Q_0$ and, moreover, each disjoint component of Q_0 is conjunctive. We have, therefore, $Q_0 \prec \uparrow(\perp_E)$. But this implies

$$f^{-1}(Q_0) \prec f^{-1}(\uparrow(\perp_E));$$

so $m \in f^{-1}(Q_0)$.

Now, $f(m)$ must be in one of the conjunctive open sets of E. Using the fact that any conjunctive subset U of a conjunctive open set V must be a non-degenerating subset of V, we conclude that $f(m)$ is maximal, thanks to Theorem 6.2.

$\square$

7 Conclusion

Maximum-separability provides an answer to the question 'when is maximal total' on dI-domains. In particular, functions on natural numbers are maximal exactly when they are 'total' in the usual sense of 'defined everywhere'. If we start from flat domains and take the maximal elements to be total, then we can take them as total in arbitrary high order domains as long as a non-maximum-separable domain does not appear as the source of a function space. However, this does not exclude the possibility of finding a proper subcategory of maximum-separable dI-domains which is also closed under the functions space construction. In fact, it is an interesting research topic to find such a category which contains infinite domains.

It should also be interesting, and perhaps more important, to look into connections between totality [1, 3, 10] and maximality in general. An immediate linkage is made difficult because in [10], for example, the notion of totality is primitive, or, given to start with, rather than derived from certain domains. We have to leave this issue as a topic for further study.

Acknowledgement I would like to thank the anonymous referees for many valuable suggestions.

8 References

[1] Abramsky, S. and Jagadeesan, R. Games and full completeness for multiplicative linear logic. *Journal of Symbolic Logic.* 1993. (to appear).

[2] Abramsky, S. and Jung, A. *Domain Theory* . *Handbook of Logic in Computer Science, vol 3.* Oxford Science Publications, 1994. (to appear).

[3] Berger, U. *Totale Objekte und Mengen in der Bereichstheorie*. Doctoral Dissertation. Department of Mathematics. Ludwig-Maximilians Universität München. 1990.

[4] Berry, G. Stable models of typed $\lambda-$calculi. *Lecture Notes in Computer Science* 62 (1978).

[5] Cartwright, R. and Demers, A. The topology of program termination. *Logic in Computer Science, 1988.*

[6] Coquand, T., Gunter, C., Winskel, G. DI-domains as a model of polymorphism. *Lecture Notes in Computer Science 298* (*Proceedings of Mathematical Foundations of Programming Semantics*). Springer, 1987.

[7] Girard, Jean-Yves. The system F of variable types, fifteen years later. *Theoretical Computer Science* 45 (1986).

[8] Girard, Jean-Yves. Linear logic. *Theoretical Computer Science* 50 (1987).

[9] Girard, Jean-Yves, Lafont, Yves, and Taylor, P. *Proofs and Types.* Cambridge Tracts in Theoretical Computer Science vol. 7. Cambridge University Press (1989).

[10] Loader, Ralph. Linear logic, totality and full completeness. *Logic in Computer Science* 1994, (to appear).

[11] Zhang, Guo-Qiang. DI-domains as prime information systems. *Information and Computation* vol 100, pp151-177 (1989).

[12] Zhang, Guo-Qiang. Stable neighborhoods. *Theoretical Computer Science* vol 93, pp143-157 (1992).

[13] Zhang, Guo-Qiang. *Logic of Domains.* Birkhauser, (1991).