Quasi-Prime Algebraic Domains[1]

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Abstract. The purpose of this paper is to introduce a new class of domains called quasi-prime algebraic domains. We show that this class is a good candidate for the purpose of denotational semantics of programming languages. This is achieved by exhibiting many constructions usually needed for denotational semantics on quasi-prime algebraic domains. In the first part we motivate the notion of quasi-primes and quasi-prime algebraic domains by studying the effect of linear functions on domains that are not prime algebraic. In the second part we study categories of quasi-prime algebraic domains. We introduce a symmetric monoidal closed category of quasi-prime algebraic domains with quasi-linear functions. We show further that with the usual Scott continuous functions as morphisms, quasi-prime algebraic domains are cartesian closed. In the third part we show the existence of a saturated quasi-prime algebraic domain, and introduce a framework for solving domain equations in quasi-prime algebraic domains. It is then possible to give, for example, a model for the un-typed lambda calculus with a reflexive quasi-prime algebraic domain.

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Part I. Quasi-Prime Algebraic Domains

1 Introduction

The most well-known domain theoretic model of linear logic is Girard's coherent spaces with linear, stable functions [6]. Other domain theoretic models of various linear logics have also been proposed. These include modoidal closed categories of dI-domains and stable event structures [22], non-stable models of linear logic [8], and prime algebraic domains [9, 19, 18].

The key notion in all these domain theoretic models of linear logics is that of *linear functions*, sometimes also called *linear maps*. Linear functions are closely related to complete primes. This is best demonstrated by a step function $[a, b]$ with $[a, b](x) = b$ for $x \sqsupseteq a$, and $[a, b](x) = \perp$ otherwise. If a non-trivial (meaning $b \neq \perp$) step function $[a, b]$ is linear, then

$$[a, b](\bigsqcup X) = \bigsqcup_{x \in X} [a, b](x)$$

for all bounded set X in the source domain. Hence, $a \sqsubseteq \bigsqcup X$ iff $\exists x \in X.a \sqsubseteq x$. In summary, this exactly says that a non-trivial step function $[a, b]$ is linear if and only if a is a complete prime. The domains mentioned in the beginning, therefore, are all prime algebraic, which also implies that they are distributive.

There are, however, other well-motivated structures which do not determine prime algebraic domains, because there is no reason to enforce distributivity. These structures include concrete data structures [11], event structures [14, 17], or even filiform structures [2]. Note also that Scott domains are not prime algebraic in general. If one ever considers those structures as possible candidates for models of linear logic, for the purpose of relating linear logic to various models of computation or just out of curiosity, one naturally would like to study the effect of linear functions acting on these structures.

After some thinking one can notice that linear function space is not a reasonable construction on non-prime algebraic domains in general, for several reasons.

- The first reason is perhaps the most important. Consider, for example, the "flat" lattice of integers (meaning only top and bottom is added to the set of integers). It is shown in [10] (for a different purpose) that the linear function space of this domain to itself, under the extensional order, is not algebraic.

- The second reason is the linear function space can become too small. There is only one non-trivial linear function from the flat lattice of integers to the Sierpinski space (a two point cpo).

- The third reason is that there may be uncountably many isolated elements in the linear function space. This is the case for the linear function space from the bottomed upside down complete binary tree (see Figure 1 on page 7) to the Sierpinski space.

There are two possible ways to solve these problems. One is to stick to the linear functions, but try to put some restrictions, not as severe as prime algebraicity, on domains so that the linear function space construction makes sense. This is the approach taken by Huth [10], who proposes the notion of *linear domains*. However, linear domains do not include many of the intuitively reasonable structures, such as as the flat integer lattice mentioned earlier.

The other approach, which is the main topic of this paper, is to generalize the notion of a linear function as far as possible. It leads us to the class of quasi-prime algebraic domains. This class includes event domains, concrete domains, and prime algebraic domains. The weaker notion of linearity introduced here is that of *quasi-linear functions*. Nevertheless, many constructions required for modeling linear logic work nicely with these non-linear functions (this is unexpected). In particular, it is still possible to break the Scott function space into two more basic constructions: the exponential, and the quasi-linear space.

We now briefly outline what quasi-prime algebraic domains are. In a prime algebraic domain, an isolated element is a complete prime if and only if there is a unique element covered by it. Correspondingly, in a Scott domain an isolated element is a quasi-prime if and only if there is a unique element covered by it. As an equivalent description, an element p is a complete prime if and only if

$$p \sqsubseteq \bigsqcup X \Rightarrow \exists x \in X.\, p \sqsubseteq x.$$

On the other hand, an element q is a quasi-prime if and only if

$$q = \bigsqcup X \Rightarrow \exists x \in X.\, p = x.$$

A Scott domain is prime algebraic if every element is the least upper bound of complete primes below (less than or equal to) it. Similarly, a Scott domain is quasi-prime algebraic if every element is the least upper bound of quasi-primes below it. As far as functions are concerned, for prime algebraic domains, a function f is linear if and only if

$$p \sqsubseteq f(x) \Rightarrow \exists r \sqsubseteq x.\, p \sqsubseteq f(r), \quad \text{where } p, r \text{ are complete primes.}$$

For quasi-prime algebraic domains, a function f is quasi-linear if and only if

$$q \sqsubseteq f(x) \Rightarrow \exists s \sqsubseteq x.\, q \sqsubseteq f(s), \quad \text{where } q, s \text{ are quasi-primes.}$$

The following table summarizes the relationships between complete primes and quasi-primes, linear functions and quasi-linear functions. For comparison, we also include isolated elements and continuous functions.

	Definition	Equivalent Characterization
Isolated Elements	$d \sqsubseteq \bigsqcup X \Rightarrow \exists x \in X.\ d \sqsubseteq x$ (X directed)	
Complete Primes	$p \sqsubseteq \bigsqcup X \Rightarrow \exists x \in X.\ p \sqsubseteq x$ (X bounded)	$\mid \{x \mid x \text{ covered by } p\} \mid = 1$ (p isolated)
Quasi-Primes	$q = \bigsqcup X \Rightarrow \exists x \in X.\ p \sqsubseteq x$	$\mid \{x \mid x \text{ covered by } p\} \mid = 1$ (p isolated)
Continuous Functions	$f(\bigsqcup X) = \bigsqcup\{f(x) \mid x \in X\}$ (X directed)	$a \sqsubseteq f(x) \Rightarrow \exists b \sqsubseteq x.\ a \sqsubseteq f(b)$ (a, b isolated)
Linear Functions	$f(\bigsqcup X) = \bigsqcup\{f(x) \mid x \in X\}$ (X bounded)	$p \sqsubseteq f(x) \Rightarrow \exists r \sqsubseteq x.\ p \sqsubseteq f(r)$ (p, r complete primes)
Quasi-Linear Functions		$q \sqsubseteq f(x) \Rightarrow \exists s \sqsubseteq x.\ q \sqsubseteq f(s)$ (q, s quasi-primes)

In this paper we show that the class of quasi-prime algebraic domains is well suited for the purpose of denotational semantics of programming languages, by exhibiting many constructions usually needed for denotational semantics on quasi-prime algebraic domains. In the remaining of the first part we introduce quasi-primes and study various properties of quasi-primes, quasi-prime algebraic domains, and quasi-linear functions. In the second part we introduce a symmetric monoidal closed category of quasi-prime algebraic domains with quasi-linear functions. We show that with the usual continuous functions as morphisms, quasi-prime algebraic domains are cartesian closed. In the third part we show the existence of a saturated quasi-prime algebraic domain, and introduce a framework for solving domain equations in quasi-prime algebraic domains. This makes it possible to give, for example, a model for the un-typed lambda calculus with a reflexive quasi-prime algebraic domain.

We believe that quasi-prime algebraic domains are an important class of domains. This is because not only they are simple to characterize, but also they provide as rich a variety of type constructions as Scott domains. In addition, quasi-prime algebraic domains better approximate many of the structures for modeling computation, such as event structures and concrete data structures, as mentioned earlier.

2 Quasi-Primes

This section introduces the notion of quasi-primes, quasi-prime algebraic domains, and related concepts. It demonstrates that quasi-primes are a natural generalization of complete primes, and quasi-prime algebraic domains a natural generalization of prime algebraic domains.

Let D be a Scott domain, i.e., a bounded complete, ω-algebraic dcpo. An element y is *covered by* x if $y \sqsubset x$ and $y \sqsubseteq z \sqsubseteq x$ implies $y = z$ or $z = x$. Write $< x >$ for the set of elements covered by x. Call $< x >$ the *lower cover* of x. A lower cover $< x >$ is *complete* if $\forall y \sqsubset x \exists x' \in < x > . y \sqsubseteq x'$. Note that if $< x >$ is complete, then either $x = \perp$, or $< x > \neq \emptyset$.

With this brief preparation, we come to the key definition of this paper.

Definition 2.1 *An isolated element q in a cpo D is a quasi-prime if there is a unique element covered by it. A Scott domain D is quasi-prime algebraic if every element of D is the least upper bound of the set of quasi-primes below (i.e., less than or equal to) it.*

For the purpose of this paper, we consider quasi-prime algebraic domains with countably many quasi-primes only. Although the notions of quasi-primes and quasi-prime algebraicity can make sense for any dcpo, here we only consider it within the class of Scott domains.

If we regard dcpos as descriptions of computational information contents, then a quasi-prime represents an atomic information increase from a unique source. If all information must be made of such atomic units, then the domain of information becomes quasi-prime algebraic.

Before we proceed further, we want to equip the reader with a wealth of simple examples of quasi-prime algebraic domains. We want to point out that although some of the proofs given below may look quite straightforward, we carefully include them here because these are basic properties on which the whole theory of quasi-prime algebraic domains is based.

Theorem 2.1 *Every finite, consistently complete cpo is a quasi-prime algebraic domain.*

Proof Let D be a finite cpo and e an element of D. We show, by mathematical induction on $\mid e \mid$, the number of elements below e, that e is the least upper bound of quasi-primes below it.

When $\mid e \mid = 1$, e must be the bottom, and there is no quasi-primes below it. By convention, however, $\perp$ is the least upper bound of the empty set. For the induction step, let us assume that all the elements e with $\mid e \mid \leq k$ are the least upper bounds of the

elements below them. Let $\mid d \mid = k+1$. If $< d >$ is a singleton, then we don't need to prove anything because d is already a quasi-prime. If, however, d is not a quasi-prime, then d is clearly the least upper bound of $< d >$. By the induction hypothesis, each element of $< d >$ is the least upper bound of the quasi-primes below it. Therefore, the same is true for d.

□

This proposition implies that the singleton cpo is a quasi-prime algebraic domain, the truth value cpo is a quasi-prime algebraic domain.

The cpo of integers is also a quasi-prime algebraic domain. With exactly the same proof, we can show the following.

Theorem 2.2 *Every finitary domain is quasi-prime algebraic. Here a finitary domain is a Scott domain in which every isolated element dominates finitely many elements.*

From Definition 2.1 it is easy to see that there are Scott domains which are not quasi-prime algebraic. The bottomed upside down complete binary tree is one such example (see Figure 1). Note that in this domain every element other than the bottom has exactly two elements immediately below it. Therefore, this domain has no quasi-primes and hence cannot be quasi-prime algebraic.

By definition, quasi-primes are isolated elements. The requirement that they are isolated plays a subtle, but important role here. Merely from the fact that $< x >$ is a singleton we cannot conclude that x is isolated. The example we have in mind is the top of the 'infinite ladder' (see Figure 2).

However, if $< x >$ is a singleton and the lower cover is also complete, then x is a quasi-prime. This is a corollary of the following lemma.

Lemma 2.1 *Let D be a Scott domain. $x \in D$ is an isolated element if $< x >$ is finite and complete. As a consequence, x is a quasi-prime if $< x >$ is a singleton, and $< x >$ is complete.*

Proof. Let $x \in D$ be such that $< x >$ is finite and complete. Since D is a Scott domain, there is a directed set Y of isolated elements such that $x = \bigsqcup Y$. Let $< x >= \{z_1, z_2, \cdots, z_m\}$. Clearly not every member of Y is dominated by z_1, which means $y_1 \not\sqsubseteq z_1$ for some $y_1 \in Y$. Similarly, for each $2 \leq i \leq m$, there is some $y_i \in Y$ such that $y_i \not\sqsubseteq z_i$. Because Y is directed, and $< x >$ is finite, there is some $y^* \in Y$ which is not dominated by any of the z_is. This y^* must be equal to x since we have no other choice: the completeness of $< x >$ guarantees that if y^* is strictly below x it must be below one of z_is. Therefore x is isolated, since y^* is.

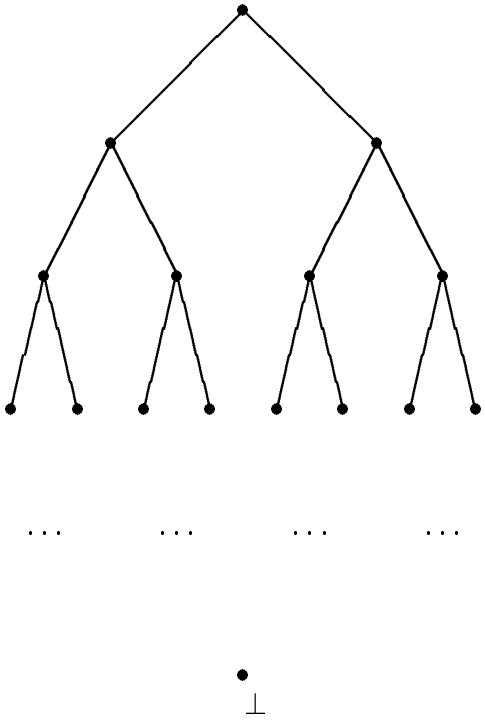


Figure 1: A Scott domain which is not quasi-prime algebraic

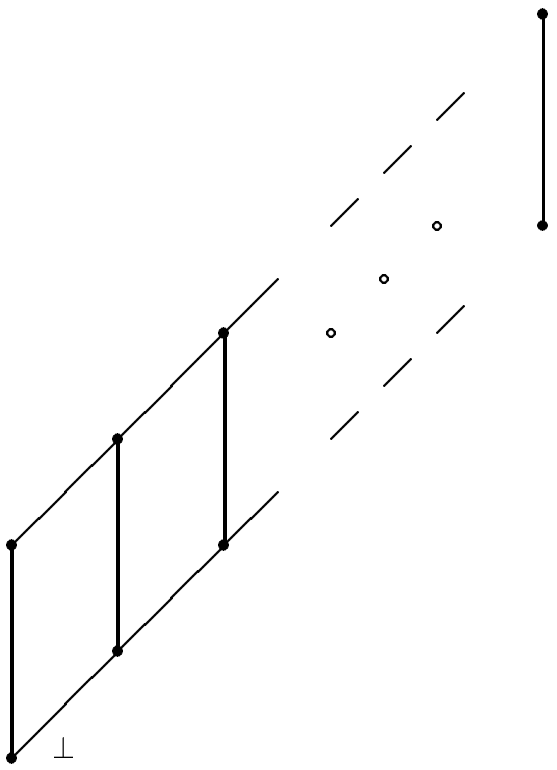


Figure 2: The infinite ladder: a unique element covered by the top

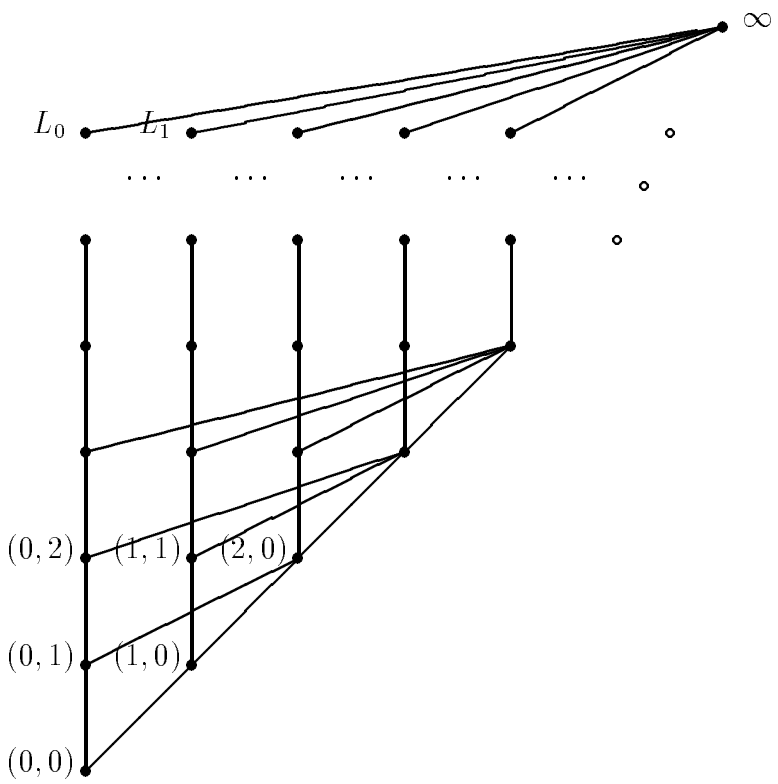


Figure 3: A non-isolated element whose lower cover is complete

□

Note that the condition "$< x >$ is finite" cannot be dropped. An example to illustrate this is given as follows (see Figure 3). Let $(D, \sqsubseteq)$ be the partial order where

$$D = \{(i,j) \mid i,j \in \omega \ \& \ j \geq i\} \cup \{L_k \mid k \in \omega\} \cup \{\infty\},$$

such that $(i,j) \sqsubseteq (i',j')$ iff either $i = i'$ and $j \leq j'$, or $j < i'$, and for each $i \in \omega$, $L_i = \bigsqcup_{j\in\omega}(i,j)$, and $\infty = \bigsqcup_{i\in\omega}(i,i)$. It can be shown that $\{L_i \mid i \in \omega\}$ is a complete set of lower cover of ∞, but ∞ is not an isolated element.

3 Complete Primes and Quasi-Primes

We now proceed to compare quasi-primes with complete primes. This will make clear why we choose the name "quasi-primes". We recall some definitions first.

Definition 3.1 *An element x of a dcpo D is a complete prime if for each set X,*

$$x \sqsubseteq \bigsqcup X \Rightarrow \exists y \in X.\ x \sqsubseteq y.$$

A Scott domain D is prime algebraic if every element of D is the join of the set of complete primes below it.

This definition does not immediately show any apparent connection between complete primes and quasi-primes. However, our next theorem reveals that a quasi-prime has a similar, but weaker, property as complete primes: for each X,

$$q = \bigsqcup X \Rightarrow \exists a \in X.\ q = a.$$

Note the only difference with the complete primes is that we replaced the the $\sqsubseteq$'s by $=$'s.

As pointed out in the previous section, simply because $< q >$ is a singleton it does not mean that q is a quasi-prime. We require that, in addition, q should be isolated. There is another point we have to be careful about. Our definition of quasi-primes did not say anything about the completeness of $< q >$. Even if $< q >$ is a singleton, we cannot automatically equate this with the property

$$\exists q^{\prec} \sqsubset q.\ (x \sqsubset q \Rightarrow x \sqsubseteq q^{\prec}),$$

i.e., every element strictly below q must be below the member in $< q >$. We need $< q >$ to be complete to make such a conclusion. However, the following lemma shows that it is safe for us to assume the completeness of $< q >$ when q is isolated.

Lemma 3.1 *Let d be an isolated element of a dcpo D. Then $< d >$ is complete.*

Proof . We use Zorn's lemma. Let $x \sqsubset d$. Consider the subset $[x, d)$, where

$$[x, d) := \{y \mid x \sqsubseteq y \sqsubset d\}.$$

With the order of D restricted to this set, we get a partial order. Let L be any chain form $[x, d)$. Because L is directed, $\bigsqcup L$ exists in D. Clearly $\bigsqcup L \sqsubseteq d$. The equality cannot occur since d is isolated. Therefore L has an upper bound in $[x, d)$. By Zorn's lemma, $[x, d)$ has a maximal element. It is easy to see that this maximal element belongs to $< d >$, and it is above x. Therefore, $< d >$ is complete.

□

We are now ready to prove the following theorem. It is interesting to note that we do not have to assume D to be a Scott domain.

Theorem 3.1 *In a bounded complete dcpo D, the following are equivalent:*

1. *q is a quasi-prime,*

2. *for every* X, $q = \bigsqcup X$ *implies* $q \in X$, *and*

3. *there exists a unique* $q^{\prec} \sqsubset q$ *such that* $y \sqsubset q$ *implies* $y \sqsubseteq q^{\prec}$.

Proof . $1 \Rightarrow 2$: This is because, by the previous lemma, $< q >$ is unique and complete.

$2 \Rightarrow 3$: Clearly $q \neq \bot$. Let $[\bot, q)$ be the set of members strictly below q, i.e.,

$$[\bot, q) := \{d \mid \bot \sqsubseteq d \sqsubset q\}.$$

This set must be directed (note that it is not empty). For any two members d_1, d_2 in $[\bot, q)$, we clearly have $d_1 \sqcup d_2 \sqsubseteq q$. But $d_1 \sqcup d_2 = q$ cannot happen because of 2 and the fact that d_1, d_2 are both strictly below q. Therefore $d_1 \sqcup d_2 \in [\bot, q)$ and $[\bot, q)$ is directed. It is now an easy step to check that

$$< q >= \{\bigsqcup[\bot, q)\}$$

and we can just take $q^{\prec}$ to be $\bigsqcup[\bot, q)$.

$3 \Rightarrow 1$: It is easy to see that $< q >= \{q^{\prec}\}$, where $q^{\prec}$ is the unique element with the property stated in 3. We only need to show that q is isolated. However, $< q >$ is complete. By Lemma 2.1, it must be isolated.

□

We do not even have to assume that D is bounded complete in the previous theorem. With almost the same proof, it can be shown that the following is true.

Theorem 3.2 *In any dcpo* D, *the following are equivalent:*

1. q *is a quasi-prime,*

2. *for every* X, $q \in \mathsf{mub}(X)$ *implies* $q \in X$, *and*

3. *there exists a unique* $q^{\prec} \sqsubset q$ *such that* $y \sqsubset q$ *implies* $y \sqsubseteq q^{\prec}$.

Here $\mathsf{mub}(X)$ *is the set of minimal upper bounds of* X.

There are two useful corollaries of Theorem 3.1.

Corollary 3.1 *In a Scott domain, every complete prime is a quasi-prime. As a consequence, we have*

1. *every complete prime is an isolated element with a unique element immediately below it, and*

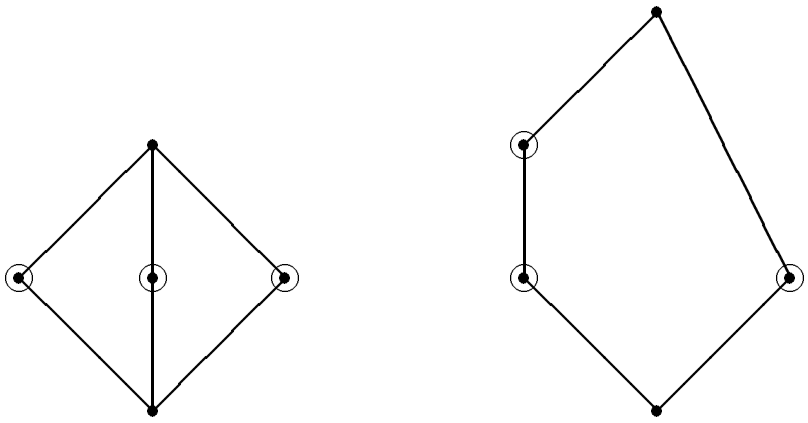


Figure 4: Two quasi-prime algebraic domains which are not prime algebraic

2. if p is a complete prime then $\exists p^{\prec} \sqsubset p.\ (y \sqsubset p \Rightarrow y \sqsubseteq p^{\prec})$.

Corollary 3.2 *Prime algebraic domains are quasi-prime algebraic.*

There are, however, quasi-prime algebraic domains which are not prime algebraic. Figure 3 contains two finite domains which are quasi-prime algebraic (by Theorem 2.1) but not prime algebraic. The circled elements are quasi-primes.

We end this section with a result relating prime algebraic domains to distributivity.

Theorem 3.3 *Let D be a quasi-prime algebraic domain. It is prime algebraic if and only if it satisfies the infinite distributive law:*

$$x \sqcap \bigsqcup X = \bigsqcup \{x \sqcap a \mid a \in X\}.$$

Proof . Only if: The proof goes by representing any element by the set of complete primes below it. The infinite distributive law then follows from the corresponding laws for sets.

If: Let D be a quasi-prime algebraic domain satisfying the infinite distributive law. It is enough to show that every quasi-prime q must actually be a complete prime. Let $q \sqsubseteq \bigsqcup X$. By the infinite distributive law, we have

$$q \;=\; q \sqcap \bigsqcup X = \bigsqcup \{q \sqcap a \mid a \in X\}.$$

By Theorem 3.1, this implies $q = q \sqcap a$ for some $a \in X$. Therefore, $q \sqsubseteq a$ for some $a \in X$. □

4 Linear Functions and Quasi-Linear Functions

Let $f : D \to E$ be a continuous function, where D and E are algebraic. If $e \sqsubseteq f(x)$ for some isolated element $e \in E$, then there is an isolated element $d \sqsubseteq x$ in D such that $e \sqsubseteq f(d)$. This follows from algebracity: by representing x as the least upper bounds of a directed set X of isolated elements, the continuity of f ensures that $e \sqsubseteq \bigsqcup\{f(y) \mid y \in X\}$, with $\{f(y) \mid y \in X\}$ again directed.

If, on the other hand, f has the property that whenever $e \sqsubseteq f(x)$ for some isolated element $e \in E$, there is an isolated element $d \in D$ such that $e \sqsubseteq f(x)$, then f must be continuous. This again follows from algebracity: given any directed set $X \subseteq D$, we have

$$f(\bigsqcup X) = \bigsqcup\{f(y) \mid y \in X\}$$

because given any isolated element e below $f(\bigsqcup X)$, there is some isolated element $d \sqsubseteq \bigsqcup X$ such that $e \sqsubseteq f(d)$. This implies $e \sqsubseteq f(y)$ for some $y \in X$, and hence, $e \sqsubseteq \bigsqcup\{f(y) \mid y \in X\}$.

Therefore, we have shown that a function f on algebraic domains is continuous if and only if

$$e \sqsubseteq f(x) \Rightarrow \exists d \sqsubseteq x.\, e \sqsubseteq f(d),$$

where d, e are isolated. By an extremely similar proof, one can show that

Proposition 4.1 *A function f on prime algebraic domains is linear if and only if*

$$q \sqsubseteq f(x) \Rightarrow \exists p \sqsubseteq x.\, q \sqsubseteq f(p),$$

where p, q are complete primes.

What is the corresponding notion of morphisms on quasi-prime algebraic domains? We take a similar property as the one for linear functions mentioned above *as the definition* for a quasi-linear function.

Definition 4.1 *Let D, E be quasi-prime algebraic domains and D^q, E^q quasi-primes of D and E, respectively. A function $f : D \to E$ is quasi-linear if*

$$\forall x \in D\ \forall e \in E^q.\, [e \sqsubseteq f(x) \ \Rightarrow\ \exists d \in D^q.\, d \sqsubseteq x\ \&\ e \sqsubseteq f(d)].$$

The following proposition provides some basic properties about quasi-linear functions. Although these are simple properties which may not be hard to verify, they are essential for us: without these properties the notion of a quasi-linear function would not make much sense.

Proposition 4.2 *Quasi-linear functions have the following properties:*

1. *every identity is quasi-linear,*

2. *composition preserves quasi-linear functions, and*

3. *quasi-linear functions are continuous.*

Proof . 1, 2 are straightforward. We give a proof for 3 here. Let $f : D \to E$ be a quasi-linear function and let X be a directed set of D. It is enough to show that

$$f(\bigsqcup X) \sqsubseteq \bigsqcup\{f(y) \mid y \in X\}.$$

The quasi-prime algebracity of D and E makes it sufficient to show that every quasi-prime below $f(\bigsqcup X)$ is also below $\bigsqcup\{f(y) \mid y \in X\}$. Let $q \sqsubseteq f(\bigsqcup X)$, where q is a quasi-prime. Because f is quasi-linear, there is a quasi-prime $p \sqsubseteq \bigsqcup X$ such that $q \sqsubseteq f(p)$. However, p is isolated. Therefore, $p \sqsubseteq x$ for some $x \in X$, which in turn implies $q \sqsubseteq f(x)$ for some $x \in X$. Hence, $q \sqsubseteq \bigsqcup\{f(y) \mid y \in X\}$.

□

Quasi-linear functions are, however, not necessarily linear. To see this, we recall the remark made in the introduction: a non-trivial step function $[a, b]$ is linear if and only if a is a complete prime. The following result shows it is as easy to find a quasi-linear, but non-linear function as to find a quasi-prime which is not a complete prime: just take a to be a quasi-prime but not a complete prime.

Proposition 4.3 *Let D, E be quasi-prime algebraic domains. A non-trivial step function $[a, b]$ is quasi-linear if and only if a is a quasi-prime.*

Proof . If: Suppose a is a quasi-prime and $q \sqsubseteq [a, b](x)$. The bottom is never a quasi-prime; therefore, $a \sqsubseteq x$. We clearly have $q \sqsubseteq [a, b](a)$.

Only if: Suppose $[a, b]$ is a non-trivial, quasi-linear function. We have $b \neq \bot_E$. By the quasi-prime algebraicty of E, there exists a quasi-prime q below b. Thus, $q \sqsubseteq [a, b](a)$. This implies $q \sqsubseteq [a, b](p)$ for some $p \sqsubseteq a$, which is only possible if $a = p$. Therefore, a is a quasi-prime.

□

The next result, whose easy proof is omitted, shows that quasi-linear functions are indeed a robust generalization of linear functions.

Proposition 4.4 *A function between prime algebraic domains is linear iff it is quasi-linear.*

5 Topological Aspects

This section discusses the topological aspects of quasi-linear functions.

Recall that a topology on a set S is a collection of subsets of S that is closed under finite intersection and arbitrary union. A set S and a topology $\Re$ on it form a topological space $(S, \Re)$. A *base* of the topology $\Re$ on S is a subset $\Re_1 \subseteq \Re$ such that every open set is the union of elements of $\Re_1$. Let $(D, \sqsubseteq)$ be a cpo. A subset O of D is said to be *Scott open* if O is upwards closed, *i.e.*, $\forall x \in O \forall y \in D$, $x \sqsubseteq y$ implies $y \in O$, and whenever $X \subseteq D$ is directed and $\bigsqcup X \in O$ we have $X \cap O \neq \emptyset$. The *Scott topology* on a cpo $(D, \sqsubseteq)$ consists of all the Scott open sets of $(D, \sqsubseteq)$, written $\mathbf{\Omega}(D)$.

Scott topology characterizes continuous functions. A function f from D to E is continuous if and only if the inverse image of any Scott open set is Scott open. Let $f, g : D \to E$ be continuous functions. $f \sqsubseteq g$ if and only if for every Scott open O of E, $f^{-1}(O) \subseteq g^{-1}(O)$.

Proposition 5.1 *$\uparrow x$ is open if and only if $x \in D^0$, where*

$$\uparrow x = \{ y \in D \mid x \sqsubseteq y \}.$$

Open sets of this form are called *prime open* since they are the complete primes in the lattice $(\mathbf{\Omega}(D), \subseteq)$.

Proposition 5.2 *The set $\{ \uparrow x \mid x \in D^0 \}$ is a base for the Scott topology of an ω-algebraic domain D.*

Thus every Scott open set is a union of open sets of the form $\uparrow x$ with $x \in D^0$. The empty set, being open, is the union of the empty collection of prime open sets.

Is there a topology which characterizes quasi-linear functions? We know that there is no topology which characterizes stable functions [21]. For a similar reason, there is no topology which characterizes either linear functions or quasi-linear functions. However, short of being closed under finite intersection, there are collections of Scott open sets which characterizes quasi-linear functions and linear functions.

Definition 5.1 *Let D be a Scott domain. A Scott open set O is called prime generated if there is a set X of complete primes of D such that*

$$O = \{y \mid \exists x \in X.\ y \sqsupseteq x\}.$$

Similarly, a Scott open set O is called quasi-prime generated if there is a set X of quasi-primes of D such that

$$O = \{y \mid \exists x \in X.\ y \sqsupseteq x\}.$$

We have the following theorem, which seems to be the best one can get concerning the topological characterization of linear and quasi-linear functions. We need a lemma first, whose proof is straightforward.

Lemma 5.1 *Let D be a quasi-prime algebraic domain. An open set O is quasi-prime generated if and only if for each y in O, there is a quasi-prime q below y, such that q is also in O.*

Theorem 5.1

1. *Let D, E be prime algebraic domains. A function $f : D \to E$ is linear if and only if the inverse image of f preserves prime generated open sets.*

2. *Let D, E be quasi-prime algebraic domains. A function $f : D \to E$ is quasi-linear if and only if the inverse image of f preserves quasi-prime generated open sets.*

Proof. As an example, we show the second conclusion.

Only if: Let $f : D \to E$ be a quasi-linear function between quasi-prime algebraic domains. Given any quasi-prime generated open set O, there is a set of quasi-primes $X \subseteq E^q$ such that

$$O = \{y \mid \exists x \in X.\, y \sqsupseteq x\},$$

by definition. We show that $f^{-1}(O)$ is again quasi-prime generated. Let $d \in f^{-1}(O)$. This means $f(d) \in O$. Therefore, $x \sqsubseteq f(d)$ for some $x \in X$. By Definition 4.1, there is a quasi-prime $q \sqsubseteq d$ such that $x \sqsubseteq f(q)$. Hence, $q \in f^{-1}(O)$. By the previous lemma, $f^{-1}(O)$ is quasi-prime generated.

If: Use Definition 4.1 again.

□

Part II. Categories

This part studies categories of quasi-prime algebraic domains. After briefly presenting a linear category of prime algebraic lattices, we introduce a monoidal closed category of quasi-prime algebraic domains with quasi-linear functions as morphisms. We also introduce a cartesian closed category of quasi-prime algebraic domains with Scott continuous functions. All our results in this section are shown with the technical tool of information systems in the sense of Scott [15].

6 Prime Algebraic Lattices: A Linear Category

This section introduces a linear category of prime algebraic lattices. This uses information systems and it serves as a bridge to the sections to follow.

We restrict ourselves to consistently complete, ω-algebraic dcpos in the rest of this paper. Thus whenever algebracity is concerned, a countable number of isolated elements is implicitly implied.

Prime algebraic domains have been discussed by a number of people [8, 9, 22, 19, 18] when domain theoretic models of linear logic were studied. In this section we show that within prime algebraic domains, the class of prime algebraic lattices is even better: they give rise to a linear category. Although the essence of these results can be derived from [8, 9, 22, 19, 18] without much difficulty, we believe our representation using prime information systems offers a pleasing easy access to these results.

Definition 6.1 *A prime information system is a structure*

$$\underline{A} = (A, \vdash)$$

where A is a countable set of tokens (propositions), and $\vdash$ is a preorder on A.

These are special kind of information systems introduced by Scott [15]. Originally, prime information systems were introduced in [19] for a representation of dI-domains. Some simplifications are made here, however:

- We do not have a consistency predicate, thus everything is consistent;
- We do not require an axiom similar to Axiom I for dI-domains;
- Further more, $\vdash$ here is a preorder ($a \vdash b$), rather than a relation between finite sets of tokens and tokens ($X \vdash a$).

The preorder $\vdash$ is intuitively understood as the entailment (that is why $\vdash$ is required to be a preorder). Thus when $a \vdash b$ we say a entails b.

Definition 6.2 *Given any subset u of tokens,*

$$\overline{u} := \{b \mid \exists a \in u.\, a \vdash b\}$$

is called the deductive closure of u. A set x is deductively closed if $x = \overline{x}$. Write $\mid \underline{A} \mid$ for the collection of deductively closed subsets of $\underline{A}$, called its elements.

Note that $\overline{u} = \overline{\overline{u}}$. Note also that ordered under inclusion, $\mid \underline{A} \mid$ is a partial order. Joins in this partial order are set unions. That is, for any subset X of elements of $\underline{A}$,

$$\bigsqcup X = \bigcup X.$$

Clearly A itself is an ideal element. The empty set is deductively closed, and it is the least element. For each single token a, $\overline{a}$ is deductively closed (for singleton sets we omit the brackets). If $\overline{a} \subseteq \bigsqcup X$, where X is a set of elements, then $a \in \bigcup X$. This implies $b \vdash a$ for some $b \in \bigcup X$. Thus $\overline{a} \subseteq x$ for some $x \in X$. Moreover, it is routine to check that, for each element x, $x = \bigsqcup\{\overline{a} \mid a \in x\}$. Thus we have the following theorem.

Theorem 6.1 *For a prime information system $\underline{A}$, $(\mid \underline{A} \mid, \subseteq)$ is a prime algebraic lattice. The complete primes are exactly of the form $\overline{a}$, with $a \in A$.*

On the other hand, each prime algebraic lattice can be associated with a prime information system. In this sense prime information systems are a representation of prime algebraic lattices.

Theorem 6.2 *Let D be a prime algebraic lattice. Then there is a prime information system $\mathcal{I}(D)$ such that $(D, \sqsubseteq)$ is isomorphic to $(\mid \mathcal{I}(D) \mid, \subseteq)$.*

Proof . Given a prime algebraic lattice D, $\mathcal{I}(D) =_{def} (D^p, \vdash)$ is a prime information system, where D^p is the set of complete primes of D, and $a \vdash b$ iff $b \sqsubseteq a$. For each $d \in D$, let

$$d \longmapsto \{e \mid e \in D^p \;\&\; e \sqsubseteq d\}.$$

This is one-one and onto, because of prime algebracity. Moreover, $d_1 \sqsubseteq d_2$ if and only if the set of complete primes below d_1 is included in the set of complete primes below d_2.

□

Let D be a prime algebraic lattice. Its dual D^{op} is again a lattice. But is it also prime algebraic? Order theoretically, this is (true but) not straightforward: a complete prime

of D may not be a complete prime of D^{op}, which can be easily seen from Theorem 3.1. The diamond with a new top is a concrete example.

However, from the point of a prime information system $(A, \vdash)$, this is intuitively easy, just take the dual of the preorder $\vdash$. Indeed, we have the following result, whose proof demonstrates the technical advantage of working with the concrete structure of a prime information system.

Theorem 6.3 *Let $(D, \sqsubseteq)$ be a prime algebraic lattice. Then*

$$D^{\mathrm{op}} \cong (\mid \mathcal{I}(D)^{\mathrm{op}} \mid, \subseteq),$$

where $\mathcal{I}(D)$ is the prime information system defined in the proof of the previous theorem, and $\mathcal{I}(D)^{\mathrm{op}}$ is the dual of the preorder $\mathcal{I}(D)$.

Proof . Let $(D, \sqsubseteq)$ be a prime algebraic lattice. Let f be a map from D to $\mid \mathcal{I}(D)^{\mathrm{op}} \mid$ such that

$$f(d) = \{t \mid t \in D^p \ \& \ t \not\sqsubseteq d\}.$$

For each $d \in D$, $\{t \mid t \in D^p \ \& \ t \not\sqsubseteq d\}$ is an element of $\mathcal{I}(D)^{\mathrm{op}}$. This because if

$$q \in \{t \mid t \in D^p \ \& \ t \not\sqsubseteq d\}$$

and $q \vdash r$ in $\mathcal{I}(D)^{\mathrm{op}}$, then r is a complete prime of D with $q \sqsubseteq r$. Hence,

$$r \in \{t \mid t \in D^p \ \& \ t \not\sqsubseteq d\}.$$

This means f is well defined.

f is clearly one-to-one. We check that it is onto. Let x be any ideal element of $\mathcal{I}(D)^{\mathrm{op}}$. Consider the element $\bigsqcup(D^p - x)$, i.e., the least upper bound of the complete primes of D not belonging to x. We claim that

$$x = f(\bigsqcup D^p - x) = \{t \mid t \in D^p \ \& \ t \not\sqsubseteq \bigsqcup(D^p - x).$$

Suppose $q \in x$, where q is a complete prime. If $q \sqsubseteq \bigsqcup(D^p - x)$, then $q \sqsubseteq r$ for some $r \in D^p - x$. But $q \in x$ and $q \sqsubseteq r$ implies $r \in x$, which is a contradiction. Therefore, we must have $q \not\sqsubseteq \bigsqcup(D^p - x)$, which implies $q \in f(\bigsqcup D^p - x)$.

Suppose, on the other hand, $q \in f(\bigsqcup D^p - x)$. We have $q \not\sqsubseteq \bigsqcup(D^p - x)$, and q is a complete prime. Clearly this means $q \notin D^p - x$, so q must be in x.

Now observe that for $d, e \in D$, $d \sqsubseteq e$ if and only if

$$\{t \mid t \in D^p \ \& \ t \not\sqsubseteq d\} \supseteq \{t \mid t \in D^p \ \& \ t \not\sqsubseteq e\}.$$

Therefore, f gives the desired isomorphism between D^{op} and $(\mid \mathcal{I}(D)^{\mathrm{op}} \mid, \subseteq)$.

□

To get a category, we have to specify the morphisms. This is based on Scott's idea of an approximable mapping.

Definition 6.3 *An approximable mapping from a prime information system $(A, \vdash_A)$ to a prime information system $(B, \vdash_B)$ is a relation $R \subseteq \mathrm{Fin}(A) \times B$ such that*

$$X \vdash_A X' \;\&\; (X', b') \in R \;\&\; b' \vdash_B b \Rightarrow (X, b) \in R.$$

Here $\mathrm{Fin}(A)$ is the collection of finite subsets of A, and $X \vdash X'$ is the abbreviation for

$$\forall a' \in X' \exists a \in X.\, a \vdash a.$$

An approximable mapping is linear if it is a relation $R \subseteq A \times B$.

The following results show that approximable mappings are closely related to continuous functions.

Theorem 6.4 *Suppose $R : \underline{A} \to \underline{B}$ is an approximable mapping. Let $Pt\,(R)$ be specified by*

$$Pt\,(R)(x) = \{\, b \mid \exists X \subseteq x.\, X\, R\, b \,\}.$$

Then $Pt\,(R)$ is a continuous function from $|\,\underline{A}\,|$ to $|\,\underline{B}\,|$. Moreover, $Pt\,(R)$ is linear if R is linear.

Proof . First we check $Pt\,(\,R\,)$ is well defined. We show that given $x \in |\,\underline{A}\,|$, $Pt\,(\,R\,)(x)$ is closed under entailment. Assume $a \in Pt\,(\,R\,)(x)$ and $a \vdash_B b$. This means $X\, R\, a$ for some $X \subseteq x$. By the previous definition, we have $X\, R\, b$, and therefore, $b \in Pt\,(\,R\,)(x)$. Since $p \in Pt\,(\,R\,)(x)$, there exists an $a \subseteq x$ such that $a\, R\, p$. By axiom 3 of Definition 3.2, $b\, R\, q$ for some $b \subseteq a$, which implies $q \in Pt\,(\,R\,)(x)$. Therefore $Pt\,(\,R\,)(x)$ is an element of $Pt\,\underline{B}$.

It is routine to check that $Pt\,(\,R\,)$ is continuous.

Let R be a linear approximable mapping and let

$$b \in Pt\, R\, (\bigcup_{i \in I} x_i).$$

Because R is linear, there is some $a \in \bigcup_{i \in I} x_i$ such that $a\, R\, b$. This implies $a \in x_i$ for some $i \in I$ and hence,

$$b \in \bigcup \{Pt\, R\, (x_i) \mid i \in I\}.$$

We have shown

$$Pt\, R\, (\bigcup_{i \in I} x_i) \subseteq \bigcup \{Pt\, R\, (x_i) \mid i \in I\},$$

which is sufficient for $Pt\,(\,R\,)$ to be linear.

□

One can show that, moreover, every linear function is determined by a linear approximable map. However, that is a consequence of Theorem 6.5 so we do not repeat it here.

Prime information systems with approximable mappings give rise to a category. The identity morphism is the approximable mapping $\{(X, b) \mid \exists a \in X. a \vdash b\}$. The composition of two approximable mappings R and S is the relational composition of $\check{}R$ with S, where

$$\check{}R =_{def} \{(X, Y) \mid \forall b \in Y.\ (X, b) \in R\}.$$

Similarly, prime information systems with linear approximable mappings constitutes a category. We write **PIS.a** and **PIS.l** for the categories of prime information systems with approximable mappings and linear approximable mappings, respectively.

Theorem 6.5 **PIS.a** *is equivalent to* **PAL.c** *and* **PIS.l** *is equivalent to* **PAL.l**. *Here* **PAL.c** *and* **PAL.l** *are categories of prime algebraic lattices with continuous functions and prime algebraic lattices with linear functions, respectively.*

Proof. By a result of MacLane [13], it is enough to show that there is full and faithful functor from **PIS.l** to **PAL.l**, and each prime algebraic lattice D is isomorphic to $(|\,\underline{A}\,|, \subseteq)$ for some prime information system $\underline{A}$. The latter is the content of Theorem 6.2, so it remains to show that there is a full and faithful functor from **PIS.l** to **PAL.l**.

The desired functor Pt maps each prime information system $\underline{A}$ to the prime algebraic lattice $(|\,\underline{A}\,|, \subseteq)$. It maps each linear approximable mapping R to the linear function

$$Pt(R) =_{def} \lambda x.\{b \mid \exists a \in x.(a, b) \in R\}.$$

Pt is a functor because it preserves identities and composition.

First we show that Pt is full. Suppose $\underline{A}$ and $\underline{B}$ are prime information systems and

$$f :|\,\underline{A}\,| \to |\,\underline{B}\,|$$

a linear function. Define a relation $R \subseteq A \times B$ by letting $(a, b) \in R$ if $b \in f(\overline{a})$. R is clearly a linear approximable mapping. Moreover, the linear function $Pt\,R$ determined by R is actually equal to f because for each element x, for each token $b \in B$, $b \in f(x)$ iff $b \in f(\overline{a})$ for some $a \in x$, according to the definition of linear functions.

Suppose $R,\ S : \underline{A} \to \underline{B}$ are linear approximable mappings such that $Pt\,R = Pt\,S$. It is easy to check that $R = S$. Therefore Pt is faithful.

□

Let $\underline{A} = (A, \vdash_A)$, $\underline{B} = (B, \vdash_B)$ be prime information systems. The following constructions can be introduced on prime information systems.

$$\begin{array}{l}
\underline{A} \otimes \underline{B} = (A \times B, \vdash_{A\otimes B}), \quad \text{where } (a,b) \vdash_{A\otimes B} (a',b') \text{ iff } a \vdash_A a' \ \& \ b \vdash_B b'; \\
\underline{A} \multimap \underline{B} = (A \times B, \vdash_{A\multimap B}), \quad \text{where } (a,b) \vdash_{A\multimap B} (a',b') \text{ iff } a' \vdash_A a \ \& \ b \vdash_B b'; \\
\underline{A} \times \underline{B} = (\{0\} \times A \cup \{1\} \times B, \vdash_{A\times B}), \quad \text{where } (i,a) \vdash_{A\times B} (j,a') \text{ iff} \\
\qquad\qquad (i = j = 0 \ \& \ a \vdash_A a') \ \text{ or } (i = j = 1 \ \& \ a \vdash_B a'); \\
\bot = (\{\bullet\}, \{(\bullet,\bullet)\}).
\end{array}$$

We follow the notation and terminology introduced in [16, 1] except $\bot$, $\top$, 0, and 1 of Girard [6]. A linear category is a symmetric monoidal closed category with all finite products and a dualizing object.

Theorem 6.6 **PIS.l** *is a linear category.* **PIS.a** *is a cartesian closed category.*

Proof. We only give a proof of the first conclusion here, since the proof for the second one is quite standard. The prime information system $(\emptyset, \emptyset)$ is both initial and terminal. The binary product is $\times$, and the tensor product is $\otimes$. One can check that $\otimes : \mathbf{PIS.l} \times \mathbf{PIS.l} \to \mathbf{PIS.l}$ is a functor satisfying the MacLane-Kelly coherence conditions [13], where the tensor unit is $\bot$. We also have

$$\underline{A} \otimes \underline{B} \multimap \underline{C} \cong \underline{A} \multimap [\underline{B} \multimap \underline{C}].$$

Since the linear approximable mappings from $\underline{A}$ to $\underline{B}$ is in one-one correspondence with the elements of $\underline{A} \multimap \underline{B}$, the above isomorphism is a key fact for monoidal closedness.

PIS.l has a dualizing object $\bot$. It is apparent that $\underline{A} \cong [[\underline{A} \multimap \bot] \multimap \bot]$. Other operators can be defined via the dualizing object. For coproduct $+$, we have

$$\underline{A} + \underline{B} =_{def} (\underline{A}^\bot \times \underline{B}^\bot)^\bot \cong \underline{A} \times \underline{B};$$

For tensorsum $\odot$, we have

$$\underline{A} \odot \underline{B} =_{def} (\underline{A}^\bot \otimes \underline{B}^\bot)^\bot \cong \underline{A} \otimes \underline{B}.$$

□

Note that gluing the bottoms and the tops of two prime algebraic domains does not give back a prime algebraic domain.

PIS.l is, moreover, a Girard category [16]. Let

$$\begin{array}{l}
\underline{A} \to \underline{B} = (\mathrm{Fin}(A) \times B, \vdash_{A\to B}), \quad \text{with } (X,a) \vdash_{A\to B} (Y,b) \ \text{ iff } Y \vdash_A X \ \& \ a \vdash_B b; \\
!\underline{A} = (\mathrm{Fin}(A), \vdash_{!A}) \ \text{ with } X \vdash_{!A} Y \ \text{ iff } X \vdash_A Y,
\end{array}$$

where $\mathrm{Fin}(A)$ stands for the set of finite subsets of A.

The function space $\to$ can be decomposed into two more primitive constructions mentioned above. This is the content of the following theorem, whose proof is straightforward, hence omitted.

Theorem 6.7 *Let $\underline{A}, \underline{B}$ be prime algebraic information systems. Then*

$$\underline{A} \to \underline{B} \cong !\underline{A} \multimap \underline{B}.$$

7 Quasi-Prime Generated Information Systems

Instead of directly working with quasi-prime algebraic domains, we introduce a representation of them as information systems. In this way, the abstract notion of a partial order will be replaced by set inclusion in a certain space. This will make the presentation and proof of the results in the next section much neater, just as we have seen for the case of prime algebraic lattices.

Definition 7.1 *An information system is a structure $\underline{A} = (A,\ Con,\ \vdash)$ where*

- *A is a countable set of propositions (tokens),*
- *Con is a collection of finite subsets of A, called the consistent sets,*
- *$\vdash \subseteq Con - \{\emptyset\} \times A$, the entailment relation,*

which satisfy

- $(X \subseteq Y \ \&\ Y \in Con) \Rightarrow X \in Con$,
- $a \in A \Rightarrow \{a\} \in Con$,
- $(X \vdash a \ \&\ X \in Con) \Rightarrow X \cup \{a\} \in Con$,
- $(a \in X \ \&\ X \in Con) \Rightarrow X \vdash a$,
- $(X \vdash Y \ \&\ Y \vdash c) \Rightarrow X \vdash c$.

Here $X \vdash Y$ is the abbreviation for $X \vdash b$ for every $b \in Y$ and $X \dashv\vdash Y$ will be used as the abbreviation for $X \vdash Y$ and $Y \vdash X$. Note that the information systems we consider here are not exactly the same as those introduced by Scott. We do not assume a distinguished element Δ, standing for *true*. Also, we require that X is non-empty when we write $X \vdash a$. This has some technical advantages, and it has the effect that the bottom element of the corresponding domain is always the empty set.

Information systems provide a representation of Scott domains. The connection between an information system and a Scott domain is via the notion of an ideal element, or just an element.

Definition 7.2 *The elements* $|\ \underline{A}\ |$*, of an information system* $\underline{A} = (\ A,\ Con,\ \vdash\ )$ *consists of subsets* x *of propositions which are*

- *consistent:* $X \subseteq^{fin} x \Rightarrow X \in Con, \quad and$
- *deductively closed:* $X \subseteq x\ \&\ X \vdash a \Rightarrow a \in x.$

We want to find a class of information systems which represent exactly the quasi-prime algebraic domains.

Definition 7.3 *Let* $\underline{A} = (A, Con, \vdash)$ *be an information system. A token* $a \in A$ *is a quasi-prime token if*

$$X \dashv\vdash a \Rightarrow a \in X.$$

We write A^q *for the set of quasi-prime tokens of* $\underline{A}$*. The information system is called quasi-prime generated iff for each* $a \in A$*, there is a finite set* $X \subseteq A^q$ *such that* $X \dashv\vdash a$*.*

Theorem 7.1 *Let* $\underline{A}$ *be an information system.* x *is a quasi-prime in* $(|\ \underline{A}\ |, \subseteq)$ *iff there exists a quasi-prime token* $a \in A$ *such that* $x = \overline{a}$*.*

We can overload the terminology a little bit and just call a quasi-prime token a quasi-prime. However, it will always be clear from the context by *quasi-primes* whether we refer to a token of an information system or an element in a domain.

Proof. Let $\overline{S}$ stand for the deductive closure of any token set S, i.e.

$$\overline{S} = \{a \mid \exists X \subseteq^{fin} S.X \vdash a\}.$$

We first show that for each quasi-prime token a, $\overline{a}$ is a quasi-prime element. For this purpose, let $y = \overline{a} - \{a' \mid a' \vdash a\}$. Then y is again an ideal element. This is because from the assumption that a is a quasi-prime token we know that for any $X \subseteq^{fin} y$ (therefore $a \vdash X$), $X \vdash a$ implies $b = a$. Now let $z \sqsubset \overline{a}$ be an ideal element. It is clear that $a \notin z$. Therefore $z \sqsubseteq y$, and y is the unique element immediately below $\overline{a}$. By Theorem 3.1, $\overline{a}$ is a quasi-prime.

Suppose x is a quasi-prime in the domain determined by a quasi-prime generated information system. That means there is a unique element y covered by x. Let $a \in x - y$. Clearly $\overline{a} \subseteq x$. If $\overline{a} \neq x$, we must have $\overline{a} \subseteq y$, since y is the unique element immediately below x, and every element strictly below x must therefore below y. This would lead to $a \in y$, contradicting our assumption $a \in x - y$. The only alternative is $\overline{a} = x$. Let $X \subseteq x$ be such that $X \vdash a$. We can also assume that propositions in X does not entail each other unless they are the same. If for each $b \in X$, $\overline{b} \subset x$, then $\overline{b} \subseteq y$ for each $b \in X$, and $\bigsqcup\{\overline{b} \mid b \in X\} \subseteq y$. This is impossible because $X \vdash a$. Therefore $\overline{b} = \overline{a}$ for some $b \in X$, which means, by antisymmetry, $b = a$.

□

Theorem 7.2 *Let $\underline{A}$ be an information system. $(|\ \underline{A}\ |, \subseteq)$ is quasi-prime algebraic iff it is quasi-prime generated.*

Proof. If: Note that the isolated elements of $(|\ \underline{A}\ |, \subseteq)$ are of the form $\overline{X}$, the deductive closure of a finite token set. From the assumption there is a finite set of quasi-prime propositions Y such that X is $\dashv\vdash$-equivalent to Y. Thus $\overline{X} = \overline{\bigcup_{b\in Y} \overline{b}}$. By Theorem 7.1, the deductive closure of each quasi-prime token is a quasi-prime. Therefore $\overline{X}$ is the join of a set of quasi-primes.

Only If: Each $a \in A$ determines an isolated element $\overline{a}$. By quasi-prime algebracity, $\overline{a}$ must be the join of a set of quasi-prime tokens, say $\{\overline{q_0}, \overline{q_1}, \cdots\}$. Therefore,

$$\overline{a} = \overline{\bigcup_{i\in\omega} \overline{q_i}}.$$

This implies the existence of a finite set Y of quasi-prime propositions from q_i's such that a is $\dashv\vdash$-equivalent to Y.

□

The following result shows that quasi-prime generated information systems can represent any quasi-prime algebraic domain.

Theorem 7.3 *For any quasi-prime algebraic domain D, there is a quasi-prime generated information system $\underline{A}$ such that $D \cong |\ \underline{A}\ |$.*

Proof. Let D be a quasi-prime algebraic domain. Define $\mathcal{I}(D) = (\mathbf{P\Omega}(D), Con, \vdash)$ where $\mathbf{P\Omega}(D)$ is the set $\{\uparrow d \mid d \text{ is a non-bottom, isolated element of } D\}$, $\vdash$ is a relation from the set of finite subsets of $\mathbf{P\Omega}(D)$ to $\mathbf{P\Omega}(D)$ such that $X \vdash a$ if $\bigcap X \subseteq a$, and $X \in Con$ if $\bigcap X \neq \emptyset$. It is easy to check that $\mathcal{I}(D)$ is an information system. We show that it is quasi-prime generated. Each quasi-prime q determines a quasi-prime proposition $\uparrow q$. This is because when

$$\bigcap_{a\in S} \uparrow a = \uparrow q$$

we have $q = \bigsqcup_{a\in S} a$. We can now activate Theorem 3.1 and arrive at the conclusion that $\uparrow a \subseteq \uparrow q$ for some $a \in S$. This is what takes for $\uparrow q$ to be a quasi-prime proposition. Similarly, the quasi-prime algebracity of $\mathcal{I}(D)$ follow from that of D. It is easy to see that D and $|\ \mathcal{I}(D)\ |$ are isomorphic.

□

8 A Monoidal Closed Category of Non-Linear Functions

We now come to one of the main focuses of the paper: a monoidal closed category of quasi-linear functions. We also introduce a cartesian closed categories of quasi-algebraic domains. It shows that quasi-prime algebraic domains can be put in service as models of intuitionistic linear logics as well as models for typed lambda-calculi. Therefore, quasi-prime algebraic domains have properties similar to Scott domains which qualify them as a mathematical foundation for programming semantics. Once again, we follow the notations proposed by Seely [16] for various linear connectives.

Here are the four major categories to be studied in this section:

QPA.ql – quasi-prime algebraic domains with quasi-linear functions,
QPA.c – quasi-prime algebraic domains with continuous functions,
QPL.ql – quasi-prime algebraic lattices with quasi-linear functions,
QPL.c – quasi-prime algebraic lattices with continuous functions.

Correspondingly, there are four categories of quasi-prime generated information systems: **QPI.ql**, **QPI.a**, **QCF.ql**, and **QCF.a**, which we introduce now.

Definition 8.1 *An approximable mapping between quasi-prime generated information systems*

$$(A, Con_A, \vdash_A)$$

and

$$(B, Con_B, \vdash_B)$$

is a relation $R \subseteq Con_A \times B^q$ such that

$$\forall R' \subseteq^{fin} R.\ \bigcup \pi_A(R') \in Con_A \Rightarrow \pi_B(R') \in Con_B,$$
$$X \vdash_A X' \ \&\ (X', b',) \in R \ \&\ b \vdash_B b' \Rightarrow (X, b) \in R.$$

Here B^q stands for the quasi-prime tokens of $\underline{B}$. An approximable mapping is quasi-linear if it is an approximable mapping R such that $(X, b) \in R$ only if $X = \{a\}$ for some quasi-prime proposition a of $\underline{A}$.

For a quasi-linear approximable mapping R, it is simpler to regard it as a relation $R \subseteq A^q \times B^q$.

One can show that there is a one-one correspondence between quasi-linear approximable mappings and quasi-linear functions. We do not expicitly state or prove it here, but that is a consequence of Theorem 8.1.

Quasi-prime generated information systems with approximable mappings form a category, denoted as **QPI.a**. Similarly, **QPI.ql** is the category of quasi-prime generated information systems with quasi-linear approximable mappings. An information system $(A, Con_A, \vdash_A)$ is *conflict-free* if every finite subset of A is a member of Con. We will simply write a conflict-free information system as $(A, \vdash_A)$, omitting the trivial consistency predicate. We write **QCF.a** for the category of conflict-free, quasi-prime generated information systems with approximable mappings, and **QCF.ql** for the category of conflict-free, quasi-prime generated information systems with quasi-linear approximable mappings.

Careful readers may want to check that indeed these are categories. This is similar to the category of prime information systems of Section 6.

Theorem 8.1 *Categorically, we have the following:*

1. **QPA.ql** *is equivalent to* **QPI.ql**,
2. **QPA.c** *is equivalent to* **QPI.a**,
3. **QPL.ql** *is equivalent to* **QCF.ql**, *and*
4. **QPL.c** *is equivalent to* **QCF.a**.

Proof. As an example, we show that **QPL.ql** is equivalent to **QCF.ql**. A functor Pt can be given from **QCF.ql** to **QPL.ql** in the following way. Pt maps each conflict-free, quasi-prime generated information system $\underline{A}$ to $(|\ \underline{A}\ |, \subseteq)$ which, by Theorem 7.2, is a quasi-prime algebraic lattice. Each quasi-linear approximable mapping R determines a quasi-linear function

$$Pt(R) =_{def} \lambda x.\{b \mid \exists a \in x.(a, b) \in R\}.$$

Furthermore, Pt is a functor because it preserves identities and composition.

By a result of MacLane [13], it is enough to show that Pt is full and faithful, and each quasi-prime algebraic lattice D is isomorphic to $(|\ \underline{A}\ |, \subseteq)$ for some conflict-free, quasi-prime generated information system $\underline{A}$. The latter is straightforward: we can take $\underline{A}$ to be $\mathcal{I}(D)$, given in Theorem 7.3. It remains to show that Pt is full and faithful.

First we show that Pt is full. Suppose $\underline{A}$ and $\underline{B}$ are conflict-free, quasi-prime generated information systems and

$$f :| \underline{A} | \rightarrow | \underline{B} |$$

a quasi-linear function. Define a relation $R \subseteq A^q \times B^q$ by letting $(a, b) \in R$ if $b \in f(\overline{a})$. R is clearly a quasi-prime linear approximable mapping. Moreover, the linear function $Pt\,R$ determined by R is actually equal to f because for each element x, for each quasi-prime

$b \in B^q$, $b \in f(x)$ iff $b \in f(\overline{a})$ for some quasi-prime $a \in x$, according to the definition of quasi-linear functions.

Suppose $R,\ S : \underline{A} \to \underline{B}$ are quasi-linear approximable mappings such that $Pt\,R = Pt\,S$. It is easy to check that $R = S$. Therefore Pt is faithful.

□

For quasi-prime generated information systems $\underline{A} = (A, Con_A, \vdash_A)$ and $\underline{B} = (B, Con_B, \vdash_B)$, we introduce the following constructions.

$\underline{A} \otimes \underline{B} = (A \times B, Con_{A\otimes B}, \vdash_{A\otimes B})$, where
$X \in Con_{A\otimes B}$ iff $\pi_A(X) \in Con_A$ & $\pi_B(X) \in Con_B$,
$X \vdash_{A\otimes B} (a,b)$ iff $\pi_A(X) \vdash_A a$ & $\pi_B(X) \vdash_B b$;
$\underline{A} \multimap \underline{B} = (A^q \times B^q, Con_{A \multimap B}, \vdash_{A \multimap B})$, where
$X \in Con_{A \multimap B}$ iff $\pi_A(X) \in Con_A \Rightarrow \pi_B(X) \in Con_B$,
$X \vdash_{A \multimap B} (a,b)$ iff $\{b' \mid \exists a'.a \vdash_A a' \ \&\ (a',b') \in X\} \vdash_B b$;
$\underline{A} \times \underline{B} = (\{0\} \times A \cup \{1\} \times B, Con_{A\times B}, \vdash_{A\times B})$, where
$X \in Con_{A\times B}$ iff $\{a \mid (0,a) \in X\} \in Con_A$ & $\{b \mid (1,b) \in X\} \in Con_B$,
$X \vdash_{A\times B} (i,p)$ iff$(i = 0$ & $\{a \mid (0,a) \in X\} \vdash_A p)$ or $(i = 1$ & $\{b \mid (1,b) \in X\} \vdash_B p)$
$\bot = (\{\bullet\}, \{\emptyset, \{\bullet\}\}, \{(\bullet,\bullet)\})$;

All these constructions preserve quasi-prime generated information systems.

Theorem 8.2 *If $\underline{A}$ and $\underline{B}$ are quasi-prime generated information systems, then $\underline{A} \otimes \underline{B}$, $\underline{A} \multimap \underline{B}$, $\underline{A} \times \underline{B}$ are quasi-prime generated information systems.*

Proof. It is easy to see that $\otimes$, $\multimap$, and $\times$ preserves information systems. We show that they preserve quasi-prime generated information systems.

$\otimes$. The quasi-primes of the tensor product $\underline{A} \otimes \underline{B}$ are $A^q \times B^q$. Because Con and $\vdash$ are defined componentwise, it is clear that $\underline{A} \otimes \underline{B}$ is a quasi-prime information system.

$\multimap$. The quasi-primes are, once again, $A^q \times B^q$. This is because by definition, $X \vdash (a,b)$ and $(a,b) \vdash X$ implies $(a',b') \vdash (a,b)$ for some $(a',b') \in X$.

$\times$. This case is simple. The quasi-primes are $\{0\} \times A^q \cup \{1\} \times B^q$.

□

Observe that there is a one-one correspondence between the set of quasi-linear approximable mappings $R : \underline{A} \to \underline{B}$ and the elements $\mid \underline{A} \multimap \underline{B} \mid$. This fact will be used in the proof of the following theorem.

Theorem 8.3 **QPI.ql** *and* **QCF.ql** *are symmetric monoidal closed categories with finite product.* **QPI.a** *and* **QCF.a** *are cartesian closed categories.*

Proof. We present the proof that **QPI.ql** is monoidal closed and **QPI.a** is cartesian closed, with some of the details are omitted.

Because the quasi-primes of $\underline{A} \otimes \underline{B}$ are exactly $A^q \times B^q$, the function $\tau : (A^q \times B^q) \times C^q \to A^q \times (B^q \times C^q)$ specified by

$$((a,b),c) \longmapsto (a,(b,c))$$

witnesses an isomorphism

$$(\underline{A} \otimes \underline{B}) \multimap \underline{C} \cong \underline{A} \multimap [\underline{B} \multimap \underline{C}].$$

In light of the remark mentioned just before the theorem, this shows there is a natural isomorphism between $\mathrm{Hom}(\underline{A} \otimes \underline{B}, \underline{C})$ and $\mathrm{Hom}(\underline{A}, \underline{B} \multimap \underline{C})$. Thus $(-) \otimes B$ is a left adjoint of $B \multimap (-)$.

It is routine to show that

$$u \in |\, \underline{A} \times \underline{B} \,| \text{ iff } \{a \mid (0,a) \in u\} \in |\, \underline{A} \,| \ \& \ \{b \mid (1,b) \in u\} \in |\, \underline{B} \,| \,.$$

Thus

$$|\, \underline{A} \times \underline{B} \,| \cong |\, \underline{A} \,| \times |\, \underline{B} \,| \,.$$

Since the cartesian product is the product in **QPA.c**, and **QPI.a** and **QPA.c** are equivalent, $\times$ is the product in **QPI.a**.

We now show that

$$\begin{aligned} \underline{A} \to \underline{B} = (&Con_A \times B^q, Con_{A \to B}, \vdash_{A \to B}), \quad \text{where} \\ &X \in Con_{A \to B} \ \text{ iff } \ \bigcup \pi_A(X) \in Con_A \Rightarrow \pi_B(X) \in Con_B, \\ &X \vdash_{A \to B} (u,b) \ \text{iff } \ \{b' \mid \exists u'.u \vdash_A u' \ \& \ (u',b') \in X\} \vdash_B b \end{aligned}$$

is the exponential. According to this construction, $X \vdash_{A \to B} (u,b)$ and $(u,b) \vdash_{A \to B} X$ implies $(u',b') \vdash_{A \to B} (u,b)$ for some $(u',b') \in X$, because b is a quasi-prime. Thus every token of $\underline{A} \to \underline{B}$ is a quasi-prime, and it is a quasi-prime generated information system. It is easy to check that $\underline{A} \times \underline{B} \to \underline{C} \cong \underline{A} \to [\underline{B} \to \underline{C}]$, since we have the structure-preserving one-one correspondence

$$(\{0\} \times X \cup \{1\} \times Y, c) \longmapsto (X,(Y,c)).$$

Therefore $(-) \times B$ is a left adjoint of $B \to (-)$.

□

As a consequence of the previous theorem, **QPA.ql** and **QPL.ql** are monoidal closed categories, and **QPA.c** and **QPL.c** are cartesian closed categories.

The decomposition $\underline{A} \to \underline{B} \cong !\underline{A} \multimap \underline{B}$ holds for both quasi-prime generated information systems and conflict-free quasi-prime generated information systems. For a quasi-prime generated information system $\underline{A}$, we have

$$\begin{aligned} !\underline{A} = (Con_A, Con_{!A}, \vdash_{!A}), \quad & \text{where} \\ X \in Con_{!A} \ \text{ iff } \ & \bigcup X \in Con_A, \\ X \vdash_{!A} v \ \text{ iff } \ & \exists u \in X.\ u \vdash_A v. \end{aligned}$$

Theorem 8.4 *Let $\underline{A}$, $\underline{B}$ be quasi-prime generated information systems. Then*

$$\underline{A} \to \underline{B} \cong !\underline{A} \multimap \underline{B}.$$

Proof . Straightforward by following the definitions.

□

Note that **QPI.ql** does not have a dualizing object. This is because $\underline{A} \multimap \bot$ is always conflict-free, but $\underline{A}$ may not be. However, **QCF.ql** does not have a dualizing object either because of the following observation.

Proposition 8.1 *Let $\underline{A}$ be a conflict-free, quasi-prime generated information system. Then*

$$(|\ \underline{A} \multimap \bot\ |, \subseteq)$$

is a prime algebraic lattice.

Proof . According to the definitions, we have

$$\begin{aligned} \underline{A} \multimap \bot = (A^q \times \{\bullet\}, \vdash_{A \multimap \bot}), \quad & \text{where} \\ X \vdash_{A \multimap \bot} (a, b) \ \text{ iff } \ & \{b' \mid \exists a'.a \vdash_A a' \ \&\ (a', b') \in X\} \vdash_\bot b. \end{aligned}$$

It is easy to see that $\underline{A} \multimap \bot$ is isomorphic to $(A^q, \vdash)$, where

$$\begin{aligned} X \vdash a \ & \Leftrightarrow \exists a' \in X.\ a \vdash_A a' \\ & \Leftrightarrow \exists a' \in X.\ a' \vdash a. \end{aligned}$$

The property

$$X \vdash a \Leftrightarrow \exists a' \in X.\ a' \vdash a$$

implies for each token $a \in A^q$, $\overline{a}$, as an ideal element, is a complete prime in $(|\ \underline{A}\ |, \subseteq)$. Therefore, $(|\ \underline{A}\ |, \subseteq)$ is a prime algebraic lattice.

□

The good news is that **QCF.a** has coproducts, which are different from products. The construction of gluing the bottoms and adding a new top to the disjoint copies of two lattices preserves quasi-prime algebraic lattices.

Part III. A Saturated Domain

We have shown that quasi-prime algebraic domains with quasi-linear functions form a monoidal closed category. The unique characteristic of the category is that the morphisms are not 'linear', as the term 'quasi-linear' suggests. This is a bit surprising, since all other known domain theoretic linear categories all use linear functions as morphisms.

However, how robust and useful the concept of quasi-prime algebraic domains is depends on whether or not they have other nice domain theoretic properties. One of the desirable properties is the existence of a universal (or even saturated) domain [7] in a certain category. The other related property to have is a framework for solving domain equations by fixed point construction, as in [12, 19]. It is the purpose of this part to establish these results for quasi-prime generated information systems, which represent quasi-prime algebraic domains.

One of the most useful results on universal domains is given in the work of Droste and Göbel [3], who introduced the Fraissé-Jónsson theorem in model theory into the area of domain theory. This makes it much easier to show the existence of certain universal domains because it reduces the existence of a saturated structure to the amalgamation property of the finite objects of a certain category.

We apply the result of Droste and Göbel for showing the existence of a saturated (universal, homogeneous) quasi-prime algebraic domain. Our main definition here is the notion of q-embeddings for quasi-prime algebraic domains. The appropriate notion of embeddings for Scott domains (call them s-embeddings) [12, 4] and for dI-domains (call them r-embeddings – 'r' for rigid) [19] are well-known . However, none of the these embeddings works for quasi-prime algebraic domains, for the following reasons:

- The s-embeddings are too general: under this embedding the colimit of an ω-chain of finite Scott domains (which are quasi-prime algebraic) need not be quasi-prime algebraic, because any Scott domain can be seen as a colimit of this kind.

- The r-embeddings are too specific: there are certain quasi-prime algebraic domains which cannot be represented as a colimit of any chain of finite Scott domains, although the r-embeddings are suitable for the I-domains (dI-domains without axiom d).

With q-embeddings we get the desired algebroidal category of quasi-prime generated information systems. The finite objects of the category are shown to have the amalgamation property. This implies the existence of a saturated quasi-prime algebraic domain.

Based on the notion of q-embeddings, we will also introduce a cpo of quasi-prime generated information systems on which various constructions are shown to induce con-

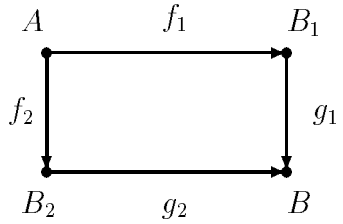
tinuous functions. This implies the existence of recursively defined quasi-prime generated information systems. Finally, we briefly illustrate how this technique can be applied to construct a model for the un-typed lambda calculus.

9 Universality and Amalgamation

A unified theory of universal objects can be found in [3]. We review the key notions of that theory in this section. The basic theorem is that in any algebroidal category in which all morphisms are monic, the existence of a universal, homogeneous object is equivalent to the amalgamation property of the (finite objects of the) category. Let us recall some of the relevant definitions now.

Let $\mathbf{C}$ be a category where all the morphisms are monic (corresponding to the intuitive notion of one-to-one). Let $\mathbf{C}_f$ be the finite objects of $\mathbf{C}$. An object U of $\mathbf{C}$ is universal in $\mathbf{C}$ (or, $\mathbf{C}$-universal) if for any object A in $\mathbf{C}$, there is an arrow $f : A \to U$. U is homogeneous (or, $\mathbf{C}_f$-homogeneous) if for any finite object A with a pair of arrows $f, g : A \to U$, there is an isomorphism $h : U \to U$ such that $f = h \circ g$. U is saturated if for any A, B of $\mathbf{C}_f$ and arrows $f : A \to U$, $g : A \to B$, there is an $h : B \to U$ such that $h \circ g = f$.

A category $\mathbf{C}$ is said to have the amalgamation property if for any arrows $f_1 : A \to B_1$, $f_2 : A \to B_2$ in $\mathbf{C}$, there are arrows $g_1 : B_1 \to B$, $g_2 : B_2 \to B$ in $\mathbf{C}$ such that the following diagram commute.



The result of Droste and Göbel is based on the notion of an algebroidal category.

Definition 9.1 *An algebroidal category is one which has the following properties:*

1. *It has an initial object,*
2. *Every object of the category is a colimit of an ω-chain of finite objects,*
3. *Every ω-chain of finite objects has a colimit, and*
4. *The number of (up to isomorphism) finite objects is countable.*

Theorem 9.1 *(Droste and Göbel) Let $\mathbf{C}$ be an algebroidal category with all morphisms monic. Let $\mathbf{C}_f$ be the full subcategory of finite objects of $\mathbf{C}$. The existence of a $\mathcal{C}$-universal,*

$\mathbf{C}_f$-homogeneous object is equivalent to the amalgamation property of $\mathbf{C}_f$, which is in turn equivalent to the existence of a $\mathbf{C}_f$ saturated object.

Note that in various categories of information systems, finite objects are often exactly those with a finite token set.

10 Q-Embeddings

To be able to apply Theorem 9.1, we need to introduce an algebroidal category of quasi-prime generated information systems. The morphisms for this category are q-embeddings.

Definition 10.1 *Let $\underline{A} = (A, Con_A, \vdash_A)$, $\underline{B} = (B, Con_B, \vdash_B)$ be quasi-prime generated information systems. A function $f : A \to B$ is a q-embedding of $\underline{A}$ into $\underline{B}$ if*

1. *f is one-to-one*;
2. $\forall X \subseteq A \forall a \in A$
   $X \in Con_A \Leftrightarrow f(X) \in Con_B$,
   $X \vdash_A a \Leftrightarrow f(X) \vdash_B f(a)$;
3. $f(A^q) \subseteq B^q$.

We remark that given a q-embedding from $\underline{A}$ to $\underline{B}$, if $f(a)$ is a quasi-prime for some $a \in A$, then a itself must be a quasi-prime. Indeed, suppose $X \dashv\vdash_A a$. Then $f(X) \dashv\vdash_B f(a)$. But $f(a)$ is a quasi-prime; so $f(a) \in f(X)$, which means $a \in X$ for f is one-one.

It is informative to show an example which is a usual embedding on information systems (i.e., that satisfies conditions 1, 2 above) [12, 4] but not a q-embedding.

Example. Let

$$\underline{A} = (\{1,3\}, Con_A, \vdash_A)$$

where Con_A includes $\{1,3\}$, and $3 \vdash_A 1$, and let

$$\underline{B} = (\{1,2,3\}, Con_B, \vdash_B)$$

where Con_B includes $\{1,2,3\}$, and $3 \vdash_B 1$, $3 \vdash_B 2$, and $\{1,2\} \vdash_B 3$. It is clear that $\underline{A}$ embeds (by identity) into $\underline{B}$ in the usual sense, but not in the sense of a q-embedding. This is because $3 \in A^q$, but $3 \notin B^q$.

Proposition 10.1 *Quasi-prime generated information systems with q-embeddings form a category, written as* $\mathbf{Q}$.

Proof. As an illustration, we check that q-embeddings compose. Let $f : \underline{A} \to \underline{B}$ and $g : \underline{B} \to \underline{C}$ be q-embeddings. It is easy to show that conditions 1 and 2 in Definition 10.1

are satisfied by $gf : A \to C$. For the third condition, we have $f(A^q) \subseteq B^q$ and $g(B^q) \subseteq C^q$. Therefore $gf(A^q) \subseteq C^q$. This shows gf is a q-embedding.

□

We now present several propositions leading to the main conclusion that **Q** is algebroidal.

Proposition 10.2 *Colimits exist in* **Q** *for ω-chains of finite information systems.*

Proof. Let $f_i : \underline{A}_i \to \underline{A}_{i+1}, i \in \omega$, be an ω-chain of finite quasi-prime generated information systems. Without loss of generality, suppose $A_i \subseteq A_{i+1}$ for $i \in \omega$. Let $\underline{A} = (A, Con, \vdash)$, where $A = \bigcup_{i\in\omega} A_i$, $\vdash = \bigcup_{i\in\omega} \vdash_i$, and $Con = \bigcup_{i\in\omega} Con_i$. It is routine to check that $\underline{A}$ is an information system. We now show that the inclusion from A_i to A is a q-embedding from $\underline{A}_i$ to $\underline{A}$. The first two conditions required of a q-embedding is easy. For the third condition, let $a \in A_i^q$ and $X \dashv\vdash a$. There is some $k \geq i$ such that $X \dashv\vdash_k a$. By Definition 10.1, $a \in A_k^q$. Therefore, $b = a$ for some $b \in X$. This implies $a \in A^q$. Hence, the inclusion is a q-embedding form $\underline{A}_i$ into $\underline{A}$. This makes it clear that $\underline{A}$ is the colimit of the chain $\underline{A}_i$, since $\underline{A}$ is quasi-prime generated.

□

Proposition 10.3 *Every quasi-prime generated information system is the colimit of an ω-chain of finite information systems in* **Q**.

Proof. With out loss of generality, let $\underline{A} = (A, Con, \vdash)$ be a quasi-prime generated information system whose quasi-primes and non-quasi-primes are both infinite. Write

$$A = \{a_i \mid i \in \omega\} \cup \{b_i \mid i \in \omega\},$$

where $A^q = \{a_i \mid i \in \omega\}$ and $A - A^q = \{b_i \mid i \in \omega\}$. Inductively, let $\underline{A}_0$ be the empty information system, and $\underline{A}_i = (A_i, Con_i, \vdash_i)$ be the finite information system such that A_i consists of $A_{i-1} \cup \{a_{i-1}\}$ plus the first element $b_j \notin A_{i-1}$, but $X \dashv\vdash b_j$ for some $X \subseteq \{a_k \mid k \leq i\}$. Let $Y \vdash_i b$ iff $Y \vdash b$ and $Y \in Con_i$ iff $Y \in Con$.

Each $\underline{A}_i$ is an information system. It is also quasi-prime generated. Suppose $X \dashv\vdash_i a_k$, where $k \leq i$. We have $X \dashv\vdash a_k$ and this means $b = a_k$ for some $b \in X$, because $a_k \in A^q$. Therefore $a_k \in A_i^q$ and $\underline{A}_i$ is quasi-prime generated. Moreover, by mathematical induction on i, one can show that $A_i^q = \{a_k \mid k \leq i\}$ for each $i \in \omega$. As a consequence, the inclusion from $\underline{A}_i$ to $\underline{A}_{i+1}$ is a q-embedding for each $i \in \omega$. It is clear that $\underline{A}$ is the colimit of the chain $\{\underline{A}_i \mid i \in \omega\}$.

□

The above propositions, together with the observation that the empty information system is initial, implies the following.

Theorem 10.1 *Quasi-prime generated information systems with q-embeddings form an algebroidal category.*

It is easy to see that corresponding to q-embeddings there is a notion of embedding-projection pairs on quasi-prime algebraic domains. These are just the usual embedding-projection pairs with the additional requirement that they preserve quasi-primes.

11 A Saturated Quasi-Prime Algebraic Domain

The purpose of this section is to show that the finite objects of the category **Q** have the amalgamation property. In light of Theorem 9.1 and Theorem 10.1, this means there exists a saturated quasi-prime generated information system. Note that our proof below follows the style of [4].

Let $\underline{A} = (A, Con_A, \vdash_A)$, $\underline{B_1} = (B_1, Con_1, \vdash_1)$ and $\underline{B_2} = (B_2, Con_2, \vdash_2)$ be finite quasi-prime generated information systems such that $f_1 : \underline{A} \to \underline{B_1}$ and $f_2 : \underline{A} \to \underline{B_2}$ are q-embeddings. By renaming the tokens, it is enough to consider the case where f_i's are inclusions (identities) and $A = B_1 \cap B_2$.

Definition 11.1 *We construct the information system* $\underline{B} = (B, Con, \vdash)$ *from* $\underline{B_1}$ *and* $\underline{B_2}$, *where*

- $B = B_1 \cup B_2$;
- $X \in Con$ *if and only if*

$$\exists Z \exists Y \supseteq X \cap B_1.\, [Y \vdash_1 Z \vdash_2 X \cap B_2] \quad or$$
$$\exists Z \exists Y \supseteq X \cap B_2.\, [Y \vdash_2 Z \vdash_1 X \cap B_1];$$

- $\vdash = \bigcup_{i \geq 0} \vdash^i$, *where* $\vdash^i$*'s are specified as follows:*

$$\vdash^0 = \{(X, a) \mid X \in Con \;\&\; \textit{either } B_1 \cap X \vdash_1 a \;\; \textit{or} \;\; B_2 \cap X \vdash_2 a\},$$
$$\vdash^{i+1} = \{(X, a) \mid \exists Y.\, X \vdash^i Y \vdash^i a\}.$$

Before getting into technical details, we would like to give the reader an intuitive feeling of what is going on. $\underline{B_1}$ and $\underline{B_2}$ are two given structures sharing a common substructure $\underline{A}$. The question is whether $\underline{B_1}$ and $\underline{B_2}$ can both be seen living in a larger structure. The larger structure must respect the already-existing relationships between $\underline{B_1}$, $\underline{B_2}$, and $\underline{A}$. In particular, $\underline{A}$ serves as a "bridge" between $\underline{B_1}$ and $\underline{B_2}$, so some X from B_1 may entail some Y from A, which may in turn entail some Z in B_2. Entailment must be transitive;

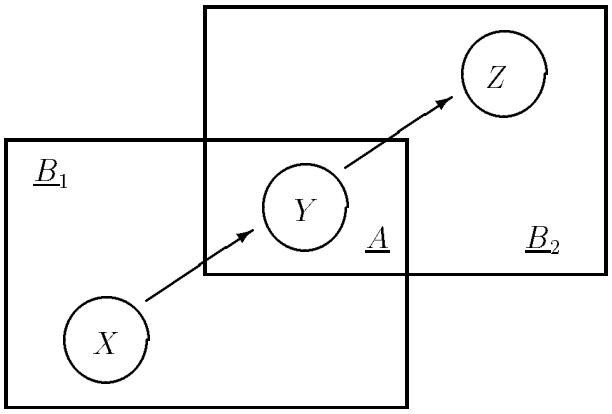


Figure 5: $\underline{A}$ acting as a bridge between $\underline{B_1}$ and $\underline{B_2}$

so in the larger structure we should have X entail Z (see Figure 5). That is why our construction of the $\vdash$ is a kind of transitive closure.

The consistency predicate Con, however, must come before $\vdash$ can be defined: the $\vdash$ relation is only defined on consistent sets. On the surface, it may look that there can be several choices for specifying Con. Letting $X \in Con$ if and only if $X \cap B_1 \in Con_1$ and $X \cap B_2 \in Con_2$, for example, may seem to be a reasonable choice. A second thought, however, reveals that this is not the case.

Example. Let $A = \{1, 2\}$, with a trivial consistency predicate and a trivial entailment relation. Let $B_1 = \{\alpha, 1, 2\}$, with $\alpha \vdash_1 1$, and let $B_2 = \{1, 2, a, b\}$ such that $\{a, b\} \notin Con_2$, and $\{1, 2\} \vdash_2 b$, $a \vdash_2 2$. It is easy to check that we have three quasi-prime generated information systems in this way, and A q-embeds into B_1 and B_2. In the bigger system to be constructed, do we want to have $\{\alpha, a\} \in Con$ because $\{\alpha\} \in Con_1$ and $\{a\} \in Con_2$? The answer is no: the given conditions $\alpha \vdash_1 1$, $a \vdash_2 2$, and $\{1, 2\} \vdash_2 b$ would imply $\{\alpha, a\} \vdash \{a, b\}$, which must be inconsistent.

□

Our original specification in Definition 11.1 seems to be the only right choice. We now check that $\underline{B}$ is indeed a quasi-prime generated information system which makes the diagram required by the amalgamation commute. This is achieved by a sequence of lemmas.

Lemma 11.1 *The structure $\underline{B}$ given in Definition 11.1 is an information system.*

Proof. We show that $\underline{B}$ satisfies the axioms of an information system.

1. $(X \subseteq Y \in Con) \Rightarrow X \in Con$: This is straightforward.

2. $b \in B \Rightarrow \{b\} \in \mathit{Con}$: If $b \in B_1 - B_2$, then

$$\{b\} \vdash_1 \emptyset \vdash_2 \{b\} \cap B_2,$$

which means, by definition, $\{b\} \in \mathit{Con}$. If $b \in B_1 \cap B_2$, then $\{b\} \vdash_1 \{b\} \vdash_2 \{b\} \cap B_2$, which again shows $\{b\} \in \mathit{Con}$.

3. $X \vdash a \Rightarrow X \cup \{a\} \in \mathit{Con}$: By mathematical induction on i, we show that each $\vdash^i$ has this property. When $i = 0$, $X \vdash^0 a$ implies two possible yet similar cases, the first being $B_1 \cap X \vdash_1 a$. Since $X \in \mathit{Con}$, there exist Y and Z such that $Y \vdash_1 Z \vdash_2 X \cap B_2$, with $Y \supseteq X \cap B_1$, or $Y \vdash_2 Z \vdash_1 X \cap B_1$, with $Y \supseteq X \cap B_2$. We have $Y \vdash_1 a$ and $Y \cup \{a\} \in \mathit{Con}_1 \subseteq \mathit{Con}$. Hence

$$Y \cup \{a\} \supseteq (X \cup \{a\}) \cap B_1.$$

If $a \notin B_2$, then

$$Y \cup \{a\} \vdash_1 Z \vdash_2 (X \cup \{a\}) \cap B_2;$$

if $a \in B_2$, then

$$Y \cup \{a\} \vdash_1 Z \cup \{a\} \vdash_2 (X \cup \{a\}) \cap B_2.$$

This shows $X \cup \{a\} \in \mathit{Con}$. For the induction step, let $X \vdash^{i+1} a$. By definition, there is some Y such that $X \vdash^i Y \vdash^i a$. Note that $\vdash^k$ has the property that $X \subseteq Y \in \mathit{Con}$ and $X \vdash^k a$ implies $Y \vdash^k a$ for each $k \geq 0$, We conclude that $X \cup Y \cup \{a\} \in \mathit{Con}$ since $X \cup Y \in \mathit{Con}$ and $X \cup Y \vdash^i a$.

4. $(X \in \mathit{Con} \ \& \ a \in X) \Rightarrow X \vdash a$: This is because either $a \in B_1$ in which case $B_1 \cap X \vdash_1 a$, or $a \in B_2$ in which case $B_2 \cap X \vdash_2 a$. we also use the fact that $\vdash^i \subseteq \vdash^{i+1}$ for each $i \geq 0$ (this can be seen by taking $Y = X$ in Definition 11.1).

5. $(X \vdash Y \ \& \ Y \vdash c) \Rightarrow X \vdash c$: Use the monotonicity of $\vdash^i$ in i.

□

Our next lemma is the key to the proofs of various results later.

Lemma 11.2 *For the information system $\underline{B}$ given in Definition 11.1, $X \subseteq B_1$ and $X \vdash Y$ imply*

$$X \vdash_1 Y \cap B_1 \quad \textit{and}$$
$$\exists Z \subseteq A.\ X \vdash_1 Z \vdash_2 Y \cap B_2.$$

Proof . By mathematical induction. When $X \vdash^0 Y$ we must have, for each $b \in Y$, either $X \vdash_1 b$, or $X \cap B_2 \vdash_2 b$. If $b \in B_1$, then in either case we have $X \vdash_1 b$; hence $X \vdash_1 Y \cap B_1$. Let

$$Z = [X \cap B_2] \cup [Y \cap B_1 \cap B_2].$$

Clearly we have $X \vdash_1 Z$. Note that

$$Y \cap B_2 = [Y \cap (B_2 - B_1)] \cup [Y \cap B_2 \cap B_1].$$

If $b \in Y \cap (B_2 - B_1)$, then we must have $X \cap B_2 \vdash_2 b$. If $b \in Y \cap B_1 \cap B_2$, then $Y \cap B_1 \cap B_2 \vdash_2 b$. Therefore,

$$X \vdash_1 Z \vdash_2 Y \cap B_2.$$

When $X \vdash^1 Y$, there is some G such that

$$X \vdash^0 G \vdash^0 Y.$$

By the previous case, there is some $Q \subseteq A$ such that

$$X \vdash_1 G \cap B_1 \ \& \ X \vdash_1 Q \vdash_2 G \cap B_2.$$

Take $R = (G \cap B_1) \cup Q$. We have $X \vdash_1 R$ and $R \subseteq B_1$. It is crucial to note that $R \vdash^0 Y$. This is because for each $b \in Y$, either $G \cap B_1 \vdash_1 b$, or $G \cap B_2 \vdash_2 b$. For the first case we have $R \vdash_1 b$, and for the latter, we have $R \cap B_2 \vdash_2 Q \vdash_2 G \cap B_2 \vdash_2 b$ and, therefore, $R \cap B_2 \vdash_2 b$. We use the result of the previous paragraph again to get

$$R \vdash_1 Y \cap B_1 \ \& \ R \vdash_1 S \vdash_2 Y \cap B_2$$

for some $S \subseteq A$. Therefore, we have

$$X \vdash_1 Y \cap B_1 \ \& \ X \vdash_1 S \vdash_2 Y \cap B_2.$$

Suppose the result holds for k, where $k \geq 1$. If $X \vdash^{k+1} Y$, then $X \vdash^k G \vdash^k Y$ for some $G \subseteq B_1 \cup B_2$. By induction hypothesis, there is some $Q \subseteq A$ such that

$$X \vdash_1 G \cap B_1 \ \& \ X \vdash_1 Q \vdash_2 G \cap B_2.$$

Take $R = (G \cap B_1) \cup Q$, which is a consistent subset of B_1. We have $R \vdash^0 G \vdash^k Y$. This means $R \vdash^0 G \vdash^{k-1} Y' \vdash^{k-1} Y$, or $R \vdash^k Y' \vdash^{k-1} Y$, for some Y'. By induction hypothesis, we have

$$R \vdash_1 Y' \cap B_1 \ \& \ R \vdash_1 S \vdash_2 Y' \cap B_2$$

for some $S \subseteq A$. Take $T = (Y' \cap B_1) \cup S$, which is a consistent subset of B_1. We have $T \vdash^0 Y' \vdash^{k-1} Y$, or $T \vdash^k Y$. We use induction hypothesis again, to get

$$T \vdash_1 Y \cap B_1 \ \& \ T \vdash_1 U \vdash_2 Y \cap B_2$$

for some $U \subseteq A$.

Therefore, we have

$$X \vdash_1 Y \cap B_1 \ \& \ X \vdash_1 U \vdash_2 Y \cap B_2.$$

□

There are a couple of ways Lemma 11.2 can be interpreted. It says that for X's from B_1, $\vdash$ is the same as $\vdash^1$. This means when X is restricted to B_1, $X \vdash Y$ if and only if $X \vdash^1 Y$. It also says that

$$X \subseteq B_1 \ \& \ X \vdash Y \Rightarrow \exists X'.\ X \vdash_1 X' \vdash^0 Y,$$

by taking X' to be $(Y \cap B_1) \cup Z$. These corollaries are sometimes more handy to use.

Lemma 11.3 *For $X \subseteq B_1$ and $a \in B_1$,*

$$X \in Con_1 \Leftrightarrow X \in Con \ \ \textit{and} \ \ X \vdash_1 a \Leftrightarrow X \vdash a.$$

Proof. If $X \in Con_1$ then by taking $Z = X \cap B_2$ we have

$$X \vdash_1 Z \vdash_2 X \cap B_2.$$

Therefore, by Definition 11.1, $X \in Con$.

Suppose $X \in Con$ and $X \subseteq B_1$. By Definition 11.1, either

$$Y \vdash_1 Z \vdash_2 X \cap B_2$$

for some $Y \supseteq X \cap B_1$, or

$$Y \vdash_2 Z \vdash_1 X \cap B_1$$

or some $Y \supseteq X \cap B_2$. The first case implies $Y \in Con_1$ and hence, so does X. For the second case, we have $Z \vdash_1 X$ and so $X \in Con_1$.

It is clear that $X \vdash_1 a$ implies $X \vdash a$. The reverse follows immediately from Lemma 11.2.

□

We now come to another important lemma which deals with quasi-primes. It shows that quasi-primes in $\underline{B_1}$ or $\underline{B_2}$ remain to be quasi-primes in $\underline{B}$.

Lemma 11.4 *Let $\underline{B}$ be the information system specified in Definition 11.1. Then every quasi-prime of its component remains to be a quasi-prime, i.e.,*

$$B_1^q \cup B_2^q \subseteq B^q.$$

Note that the equality $B_1^q \cup B_2^q = B^q$ follows easily from this lemma.

Proof . Let $X \dashv\vdash a$ for $a \in B_1^q$. We show, by mathematical induction on i, that

$$X \dashv\vdash^i a \Rightarrow a \in X$$

for $i \geq 0$.

Let $i = 0$. We have $X \vdash^0 a$ and $a \vdash^0 X$. By Lemma 11.2, there is some $Z \subseteq A$ such that

$$a \vdash_1 X \cap B_1 \;\&\; a \vdash_1 Z \vdash_2 X \cap B_2.$$

Note that $X \vdash^0 a$ means either $X \cap B_1 \vdash_1 a$, or $X \cap B_2 \vdash_2 a$. The first case implies $X \cap B_1 \dashv\vdash_1 a$, so $a \in X \cap B_1$ since a is a quasi-prime of $\underline{B_1}$. For the second case, we have $Z \vdash_2 a$, which implies $Z \vdash_1 a$ because both a and Z are form B_1. In this case we have $Z \dashv\vdash_1 a$ and hence $a \in Z$. Note that a is a quasi-prime of $\underline{A}$, which implies it is also a quasi-prime of $\underline{B_2}$. However, $X \cap B_2 \dashv\vdash_2 a$, so $a \in X \cap B_2$.

Now consider $X \dashv\vdash^k a$ for some $k \geq 1$. Since $a \vdash X$ and $a \in B_1$, by Lemma 11.2 we have

$$a \vdash_1 X \cap B_1 \;\&\; a \vdash_1 Z_1 \vdash_2 X \cap B_2$$

for some $Z_1 \subseteq A$. Let $Y_1 = (X \cap B_1) \cup Z_1$. By repeated application of Lemma 11.2, we have

$$\begin{aligned} a \vdash_1 Y_1 \vdash^0 X \vdash a \;&\Rightarrow a \vdash_1 Y_1 \vdash a \\ &\Rightarrow \exists Y_2 \subseteq B_1.\, a \vdash_1 Y_1 \vdash_1 Y_2 \vdash^0 a \\ &\Rightarrow a \vdash_1 Y_1 \vdash_1 Y_2 \vdash_1 a \\ &\Rightarrow a \in Y_1. \end{aligned}$$

This means either $a \in X \cap B_1$, or $a \in Z_1$. If $a \in X \cap B_1$, we are done; so assume $a \in Z_1$. We have

$$a \vdash_1 X \cap B_1 \;\text{ and }\; a \vdash_2 Z_1 \vdash_2 X \cap B_2.$$

Therefore, $a \vdash^0 X$.

Induction Step: $X \vdash^k a$ implies $X \vdash^{k-1} G \vdash^{k-1} a$ for some G. Hence, $a \vdash^k G \vdash^{k-1} a$, and this means, by applying Lemma 11.2,

$$a \vdash_1 G \cap B_1 \;\&\; a \vdash_1 Z_2 \vdash_2 G \cap B_2$$

for some $Z_2 \subseteq A$. Let $Y_3 = (G \cap B_1) \cup Z_2$. We have $Y_3 \subseteq B_1$ and $a \vdash_1 Y_3 \vdash^0 G$. By repeated application of Lemma 11.2, we get

$$\begin{aligned} a \vdash_1 Y_3 \vdash^0 G \vdash a \;&\Rightarrow a \vdash_1 Y_3 \vdash a \\ &\Rightarrow \exists Y_4 \subseteq B_1.\, a \vdash_1 Y_3 \vdash_1 Y_4 \vdash^0 a \\ &\Rightarrow a \vdash_1 Y_3 \vdash_1 Y_4 \vdash_1 a \\ &\Rightarrow a \in Y_3. \end{aligned}$$

Either $a \in G \cap B_1$, or $a \in Z_2$.

If $a \in G \cap B_1$, then $X \vdash^{k-1} a$. Thus we have $X \dashv\vdash^{k-1} a$, and we apply the induction hypothesis to get $a \in X$.

If $a \in Z_2$, then

$$[a \vdash_1 G \cap B_1] \text{ and } [a \vdash_2 Z_2 \vdash_2 G \cap B_2] \text{ implies } a \vdash^0 G.$$

Therefore, $a \dashv\vdash^{k-1} G$, which implies $a \in G$, by induction hypothesis. This means $X \dashv\vdash^{k-1} a$ and we apply the induction hypothesis again to get $a \in X$.

□

As a corollary of the previous lemma, we have

Theorem 11.1 *$\underline{B}$ is a quasi-prime generated information system. Moreover, both $\underline{B}_1$ and $\underline{B}_2$ q-embed into $\underline{B}$.*

In summary, we have proved

Theorem 11.2 *The finite objects of the category* **Q** *has the amalgamation property.*

As a consequence, we have shown the following.

Theorem 11.3 *There exists a saturated quasi-prime algebraic domain.*

12 A CPO of Quasi-Prime Generated Information Systems

One of the purposes of a universal domain is to ensure the existence of solutions to domain equations for denotational semantics of programming languages. Another way to achieve the same goal is to introduce a cpo of quasi-prime generated information systems using a substructure relation. Various constructions on information systems can be shown to induce continuous functions on the cpo. The existence of the least fixed point for continuous functions then guarantees the existence of solution to domain equations. The advantage of this approach is that the solution is up to equality, rather than up to an isomorphism. We describe this approach now, which uses the idea first introduced in [12].

Definition 12.1 *Let $\underline{A} = (A, Con_A, \vdash_A)$ and $\underline{B} = (B, Con_B, \vdash_B)$ be quasi-prime generated information systems. $\underline{A} \trianglelefteq \underline{B}$ if*

1. $A \subseteq B$
2. $X \in Con_A \iff X \subseteq A \;\&\; X \in Con_B$,
3. $X \vdash_A a \iff X \cup \{a\} \subseteq A \;\&\; X \vdash_B a$, *and*
4. $A^q \subseteq B^q$.

When $\underline{A} \trianglelefteq \underline{B}$ we call $\underline{A}$ a subsystem of $\underline{B}$. Our definition of subsystem is similar to that of Larsen and Winskel [12], with item 4 the only extra requirement. Note that for quasi-prime generated information system $\underline{A}$ and $\underline{B}$, if $A = B$ and $\underline{A} \trianglelefteq \underline{B}$, then $\underline{A} = \underline{B}$.

The relation $\trianglelefteq$ is a complete partial order on the *class* of quasi-prime generated information systems.

Theorem 12.1 *The relation $\trianglelefteq$ is a partial order with the least element*

$$\underline{\bot} = (\,\emptyset,\ \{\,\emptyset\,\},\ \emptyset\,).$$

If $\underline{A_0} \trianglelefteq \underline{A_1} \trianglelefteq \cdots \trianglelefteq \underline{A_i} \trianglelefteq \cdots$ is an increasing chain of stable information systems where $\underline{A_i} = (\,A_i,\ Con_i,\ \vdash_i\,)$, then their least upper bound is

$$\bigcup_i \underline{A_i} = \left(\bigcup_i A_i,\ \bigcup_i Con_i,\ \bigcup_i \vdash_i\right).$$

Proof. Straightforward.

□

Write **CPO** for the class of quasi-prime generated information systems under $\trianglelefteq$. **CPO** is not a cpo in the usual sense simply because they are not a set but a class. The subsystem relation $\trianglelefteq$ can be easily extended to $n-$tuples in the same way as described in [19]. A useful observation is that a unary operation F is continuous iff it is monotonic with respect to $\trianglelefteq$ and continuous on proposition sets, *i.e.* for any $\omega-$chain

$$\underline{A_1} \trianglelefteq \underline{A_2} \cdots \trianglelefteq \underline{A_i} \trianglelefteq \cdots,$$

each proposition of $F\,(\bigcup_i \underline{A_i}\,)$ is a proposition of $\bigcup_i F\,(\,\underline{A_i}\,)$.

Many constructions can be introduced on quasi-prime generated information systems such as sum, product, lifting, function space, and even quasi-linear function space, as we have seen in Part II.

In the rest of the section we illustrate that function space $\rightarrow$ corresponds to a continuous operation

$$\rightarrow\colon\ \mathbf{CPO}^2 \rightarrow \mathbf{COP}.$$

Other constructions can be shown, in a similar way, to induce continuous operations on **CPO**.

Recall that the function space construction is given as follows.

$$\begin{aligned}
&\underline{A} \rightarrow \underline{B} = (Con_A \times B^q, Con_{A\rightarrow B}, \vdash_{A\rightarrow B}), \quad \text{where}\\
&\qquad X \in Con_{A\rightarrow B} \ \text{ iff } \ \bigcup \pi_A(X) \in Con_A \Rightarrow \pi_B(X) \in Con_B,\\
&\qquad X \vdash_{A\rightarrow B} (u, b) \ \text{iff } \ \{b' \mid \exists u'.u \vdash_A u' \,\&\, (u', b') \in X\} \vdash_B b
\end{aligned}$$

We have shown in Part II that $\rightarrow$ preserves quasi-prime generated information systems. $\rightarrow$ is monotonic in its first argument. Suppose $\underline{A} \trianglelefteq \underline{A'}$. Write

$$\underline{C} = (\, C,\, Con,\, \vdash\,) = [\underline{A} \rightarrow \underline{B}]$$

and

$$\underline{C'} = (\, C',\, Con',\, \vdash'\,) = [\underline{A'} \rightarrow \underline{B}].$$

We check condition 4 in Definition 12.1, to show that $\underline{C} \trianglelefteq \underline{C'}$. However, this is trivial because every token of the function space is a quasi-prime.

Let

$$\underline{A_0} \trianglelefteq \underline{A_1} \trianglelefteq \cdots \trianglelefteq \underline{A_i} \trianglelefteq \cdots$$

be a chain of quasi-prime generated information systems. Let (u, a) be a token of $[(\bigcup_i \underline{A_i}) \rightarrow \underline{B}]$. Then clearly (u, a) is a token of $[\underline{A_j} \rightarrow \underline{B}]$ for some j, and thus is a token of $\bigcup_i [\underline{A_i} \rightarrow \underline{B}\,]$. From this we can deduce that $\rightarrow$ is continuous in its first argument. By a similar but easier proof we get that $\rightarrow$ is continuous in its second argument hence it is continuous.

As an application, we illustrate how to find a solution to the equation

$$X = X_{\uparrow} \rightarrow X$$

within quasi-prime generated information systems. Here $(\quad)_{\uparrow}$ is the lifting construction specified as follows. Its use here makes sure that a non-trivial solution is obtained.

Let $\underline{A} = (\, A, Con, \vdash)$ be a quasi-prime generated information system. Define the *lift* of $\underline{A}$ to be $\underline{A}_{\uparrow} = (\, A',\, \vdash')$, where

- $A' = (\, \{\, 0\, \} \times A\,) \cup \{\, 0\,\}$,
- $X \vdash' Y \Leftrightarrow [\, 0 \in Y \;\text{ or }\; \{\, c \mid (0,\, c\,) \in X\,\} \vdash_{\underline{A}} \{\, b \mid (0,\, b\,) \in Y\,\}\,]$.

It is easy to show that lifting preserves quasi-prime generated systems. It is an operation which, given a structure, produces a new one by joining a new token weaker than all the old ones. One can easily check that it gives a continuous operation.

Since the composition of continuous functions remains continuous, and any continuous function $F(x, y)$ of two variables gives rise to a continuous function $F(x,\, x)$ of one variable, the operation

$$X \;\longmapsto\; [X_{\uparrow} \rightarrow X]$$

is a continuous operation on quasi-prime generated information systems. It has a least fixed point

$$\underline{A} = \underline{A}_{\uparrow} \rightarrow \underline{A}.$$

This can be used as a model for the un-typed lambda calculus. However, we do not have space to get into more details here.

13 Concluding Remarks

We have introduced a new class of domains called quasi-prime algebraic domains. This class is a good candidate for the purpose of denotational semantics: it has a rich type structure. We have shown that quasi-prime algebraic domains have many nice features, such as they form a cartesian closed category, a monoidal closed category, with appropriate morphisms. Moreover, there is a saturated quasi-prime algebraic domain and it is possible to solve domain equations in quasi-prime algebraic domains.

Information systems are an important technical tool which have helped us achieving all these. They not only have helped us resolving many technically difficult issues in an elegant way, but also have helped us presenting the results conciselly. Although some of our results may look quite straightforward because of this, we think this is a virtue: simple is better.

We remark that a dual concept of quasi-primes was mentioned in [5] (pages 92-93), called the *completely irreducible elements*. Quasi-primes were introduced in [20] without the author's knowledge of that. However, our objectives and motivations are totally different. Furthermore, we have gone far beyond just a particular class of elements in a lattice. We consider domains generated by these elements, and we consider categories of quasi-prime algebraic domains. More important, we have introduced quasi-linear functions which are the corresponding morphisms for quasi-primes.

We would like to comment that quasi-prime algebraic domains are motivated from theoretical properties of certain constructions on domains. Perhaps a more appropriate way to motivate a new class of domains is to show adequacy or even full abstraction results for a proper programming language with a suitable operational semantics. Although the latter is probably more desirable, it is impossible or even not beneficial to have every body follow the same line of thought.

As far as future work is concerned, there is a host of issues to be explored. Within quasi-prime algebraic domains, the most immediate topic is to consider stable functions as morphisms (this is to appear in a different paper). Other non-distributive structures within Scott domains can be reinvestigated in light of quasi-primes. Adequacy and full abstruction problems may be studied using quasi-prime algebraic domains as the semantic space.

The notion of a quasi-prime makes sense beyond Scott domains. Our Theorem 3.2 is deliberately suggesting this. Categories of quasi-prime SFP domains, and quasi-prime L-domains may be introduced, for example. However, how useful these results will be, I don't know (I sometimes tend to think that satisfying one's curiosity is also a worthy cause).

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