Quasi-Prime Algebraic Domains

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The purpose of this paper is to introduce a new class of domains called quasi-prime algebraic domains. We show that this class is a good candidate for the purpose of denotational semantics of programming languages. This is achieved by exhibiting many constructions usually needed for denotational semantics on quasi-prime algebraic domains. In the first part we motivate the notion of quasi-primes and quasi-prime algebraic domains by studying the effect of linear functions on domains that are not prime algebraic. In the second part we study categories of quasi-prime algebraic domains by representing them as irreducible information systems. We introduce a symmetric monoidal closed category of quasi-prime algebraic domains with quasi-linear functions. We show further that with the usual Scott continuous functions as morphisms, quasi-prime algebraic domains are cartesian closed. In the third part we show the existence of a saturated quasi-prime algebraic domain, and introduce a framework for solving domain equations in quasi-prime algebraic domains. It is then possible to give, for example, a model for the un-typed lambda calculus with a reflexive quasi-prime algebraic domain.

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Part I. Quasi-Prime Algebraic Domains

1 Introduction

The most well-known domain theoretic model of linear logic is Girard's coherent spaces with linear, stable functions [6]. Other domain theoretic models of various linear logics have also been proposed. These include monoidal closed categories of dI-domains and stable event structures [21], non-stable models of linear logic [7], and prime algebraic domains [8,18,17].

The key notion in all these domain theoretic models of linear logics is that of *linear functions*, sometimes also called *linear maps*. Linear functions are closely related to complete primes. This is best demonstrated by a step function $[a, b]$ with $[a, b](x) = b$ for $x \sqsupseteq a$, and $[a, b](x) = \perp$ otherwise. If a non-trivial (meaning $b \neq \perp$) step function $[a, b]$ is linear, then

$$[a, b](\bigsqcup X) = \bigsqcup_{x \in X} [a, b](x)$$

for all bounded set X in the source domain. Hence, $a \sqsubseteq \bigsqcup X$ if and only if $a \sqsubseteq x$ for some x in X. Therefore, a non-trivial step function $[a, b]$ is linear if and only if a is a complete prime. The domains mentioned at the beginning, therefore, are all prime algebraic (which also implies that they are distributive).

There are, however, other well-motivated structures which do not determine prime algebraic domains, because distributivity is not enforced on them. These structures include concrete data structures [10], event structures [13,16], or even filiform structures [2]. Moreover, Scott domains are not prime algebraic in general. If one ever considers those structures as possible candidates for models of linear logic, for the purpose of relating linear logic to various models of computation or just out of curiosity, one naturally would like to study the effect of taking linear functions as morphisms on these structures.

After some thinking one can notice that the linear function space is not a reasonable construction on non-prime algebraic domains in general, for a couple of reasons.

- The first reason is perhaps the most important. Consider, for example, the "flat" lattice of integers (meaning only top and bottom are added to the set of integers). It is shown in [9] that the linear function space of this domain to itself, under the extensional order, is not algebraic.
- The second reason is that there may be uncountably many isolated elements in the linear function space. This is the case for the linear function space from the bottomed upside down complete binary tree (see Figure 1

on page 8) to the Sierpinski space.

There are two possible ways to solve these problems. One is to stick to linear functions while putting some restrictions, not as severe as prime algebraicity, on domains so that the linear function space construction makes sense. This is the approach taken by Huth [9], who proposes the notion of *linear domains*. Note that linear domains do not include the flat integer lattice mentioned earlier, among other domains.

The other approach, which is the main topic of this paper, is to generalize the notion of a linear function as much as possible. This leads to the class of quasi-prime algebraic domains, which include event domains, concrete domains, and prime algebraic domains. The weaker notion of linearity introduced here is that of *quasi-linear functions*. Nevertheless, many constructions required for modeling linear logic work nicely with these non-linear functions (this is unexpected). In particular, it is still possible to break the Scott function space into two more basic constructions: the exponential, and the quasi-linear space (this is also not expected).

We now briefly explain what quasi-prime algebraic domains are. For this purpose, it is helpful to compare quasi-primes with complete primes, a standard notion in domain theory or lattice theory. In a prime algebraic domain, an isolated element is a complete prime if and only if there is a unique element covered by it. Correspondingly, in a Scott domain, an isolated element is a quasi-prime if and only if there is a unique element covered by it. As an equivalent description, an element p is a complete prime if and only if

$$p \sqsubseteq \bigsqcup X \Rightarrow \exists x \in X.\, p \sqsubseteq x;$$

On the other hand, an element q is a quasi-prime if and only if

$$q = \bigsqcup X \Rightarrow \exists x \in X.\, p = x.$$

A Scott domain is prime algebraic if every element is the least upper bound of complete primes below (less than or equal to) it. Similarly, a Scott domain is quasi-prime algebraic if every element is the least upper bound of quasi-primes below it. As far as functions are concerned, for prime algebraic domains, a function f is linear if and only if

$$p \sqsubseteq f(x) \Rightarrow \exists r \sqsubseteq x.\, p \sqsubseteq f(r), \quad \text{where } p, r \text{ are complete primes.}$$

For quasi-prime algebraic domains, a function f is quasi-linear if and only if

$$q \sqsubseteq f(x) \Rightarrow \exists s \sqsubseteq x.\, q \sqsubseteq f(s), \quad \text{where } q, s \text{ are quasi-primes.}$$

The following table summarizes the relationships between complete primes

and quasi-primes, linear functions and quasi-linear functions. For comparison, we also include items for isolated elements and continuous functions.

	Definition	Equivalent Condition
Finite element	$d \sqsubseteq \bigsqcup X \Rightarrow \exists x \in X.\ d \sqsubseteq x$ (X directed)	
Complete prime	$p \sqsubseteq \bigsqcup X \Rightarrow \exists x \in X.\ p \sqsubseteq x$ (X bounded)	$\mid \{x \mid x \text{ covered by } p\} \mid = 1$ (p isolated)
Quasi prime	$q = \bigsqcup X \Rightarrow \exists x \in X.\ p \sqsubseteq x$	$\mid \{x \mid x \text{ covered by } p\} \mid = 1$ (p isolated)
Continuous fun.	$f(\bigsqcup X) = \bigsqcup\{f(x) \mid x \in X\}$ (X directed)	$a \sqsubseteq f(x) \Rightarrow \exists b \sqsubseteq x.\ a \sqsubseteq f(b)$ (a, b isolated)
Linear fun.	$f(\bigsqcup X) = \bigsqcup\{f(x) \mid x \in X\}$ (X bounded)	$p \sqsubseteq f(x) \Rightarrow \exists r \sqsubseteq x.\ p \sqsubseteq f(r)$ (p, r complete primes)
Quasi-linear fun.		$q \sqsubseteq f(x) \Rightarrow \exists s \sqsubseteq x.\ q \sqsubseteq f(s)$ (q, s quasi-primes)

In this paper we show that the class of quasi-prime algebraic domains is well suited for the purpose of denotational semantics of programming languages by exhibiting many constructions usually needed for denotational semantics on quasi-prime algebraic domains. In the remaining of the first part we introduce quasi-primes and study various properties of quasi-primes, quasi-prime algebraic domains, and quasi-linear functions. In the second part we introduce a symmetric monoidal closed category of quasi-prime algebraic domains with quasi-linear functions. We show that with the usual continuous functions as morphisms, quasi-prime algebraic domains are cartesian closed. In the third part we show the existence of a saturated quasi-prime algebraic domain, and introduce a framework for solving domain equations in quasi-prime algebraic domains. This makes it possible to give, for example, a model for the un-typed lambda calculus with a reflexive quasi-prime algebraic domain.

We work with quasi-prime algebraic domains through the representation of them as *irreducible information systems*. These are a special class of information systems without redundant tokens, with non-redundancy in the sense that the informational content of a token cannot be recaptured by other tokens in the system. Irreducible information systems not only serve as an elegant technical tool, they also provide an illuminating connection between quasi-prime algebraicity and the existence of the most efficient representation of a

Scott-domain.

2 Quasi-Primes

This section introduces the notion of quasi-primes, quasi-prime algebraic domains, and related concepts. It demonstrates that quasi-primes are a natural generalization of complete primes, and quasi-prime algebraic domains a natural generalization of prime algebraic domains.

Let D be a Scott domain, *i.e.*, a bounded complete, ω-algebraic dcpo. An element y is *covered by* x if $y \sqsubset x$ and $y \sqsubseteq z \sqsubseteq x$ implies $y = z$ or $z = x$. Write $< x >$ for the set of elements covered by x. Call $< x >$ the *lower cover* of x. A lower cover $< x >$ is *complete* if for any element y strictly below x, y is below some element x' in the lower cover of x. Note that if $< x >$ is complete, then either $x = \perp$, or $< x > \neq \emptyset$.

With this brief preparation, we come to the key definition of this paper.

Definition 1 *An isolated element q in a cpo D is a quasi-prime if there is a unique element covered by it. A Scott domain D is quasi-prime algebraic if every element of D is the least upper bound of the set of quasi-primes below (*i.e.*, less than or equal to) it.*

For the purpose of this paper, we consider quasi-prime algebraic domains with countably many quasi-primes only. Although the notions of quasi-primes and quasi-prime algebraicity can make sense for any dcpo, we only consider them in this paper within the class of Scott domains.

If we regard dcpos as descriptions of computational information contents, then a quasi-prime represents an atomic information increase from a unique source. If all information must be made of such atomic units, then the domain of information becomes quasi-prime algebraic.

Before proceeding further, we want to equip the reader with a wealth of simple examples of quasi-prime algebraic domains. We want to point out that although some of the proofs given below may look quite straightforward, we carefully include them here because these are basic properties on which the whole theory of quasi-prime algebraic domains is based.

Theorem 2 *Every finite, consistently complete cpo is a quasi-prime algebraic domain.*

PROOF. Let D be a finite cpo and e an element of D. We show, by mathe-

matical induction on $\mid e \mid$, the number of elements below e, that e is the least upper bound of quasi-primes below it.

When $\mid e \mid = 1$, e must be the bottom, and there is no quasi-primes below it. By convention, however, $\perp$ is the least upper bound of the empty set. For the induction step, let us assume that all the elements e with $\mid e \mid \leq k$ are the least upper bounds of the elements below them. Let $\mid d \mid = k+1$. If $< d >$ is a singleton, then we don't need to prove anything because d is already a quasi-prime. If, however, d is not a quasi-prime, then d is clearly the least upper bound of $< d >$. By the induction hypothesis, each element of $< d >$ is the least upper bound of the quasi-primes below it. Therefore, the same is true for d.

□

This proposition implies that the singleton cpo is a quasi-prime algebraic domain, the truth value cpo is a quasi-prime algebraic domain. Of course, as the name suggests, prime algebraic domains are quasi-prime algebraic.

The cpo of integers is also a quasi-prime algebraic domain. With exactly the same proof, we can show the following.

Theorem 3 *Every finitary domain is quasi-prime algebraic. Here a finitary domain is a Scott domain in which every isolated element dominates finitely many elements.*

From Definition 1, it is easy to see that there are Scott domains which are not quasi-prime algebraic. The bottomed upside down complete binary tree is one such example (see Figure 1). Note that in this domain every element other than the bottom has exactly two elements immediately below it. Therefore, this domain has no quasi-primes and hence cannot be quasi-prime algebraic.

By definition, quasi-primes are isolated elements. The requirement that they are isolated plays a subtle, but important role. Merely from the fact that $< x >$ is a singleton we cannot conclude that x is isolated. The example we have in mind is the top of the "infinite ladder" (see Figure 2).

However, if $< x >$ is a singleton and the lower cover is also complete, then x is a quasi-prime. This is a corollary of the following lemma.

Lemma 4 *Let D be a Scott domain. $x \in D$ is an isolated element if $< x >$ is finite and complete. As a consequence, x is a quasi-prime if $< x >$ is a singleton, and $< x >$ is complete.*

PROOF. Let $x \in D$ be such that $< x >$ is finite and complete. Since D is a

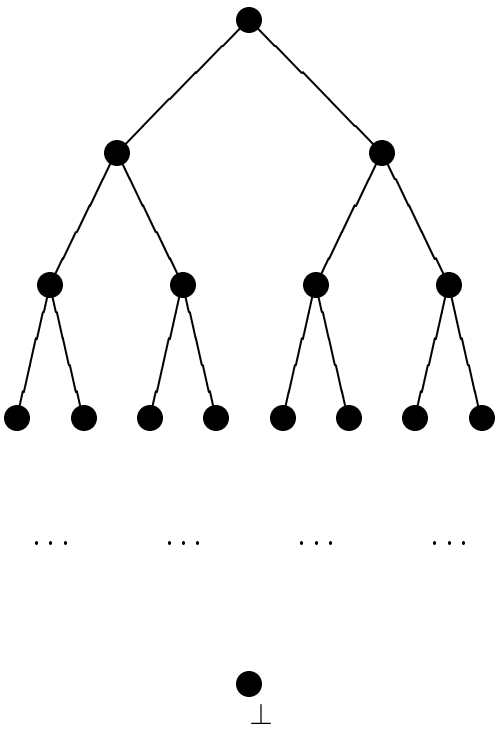


Fig. 1. A Scott domain which is not quasi-prime algebraic

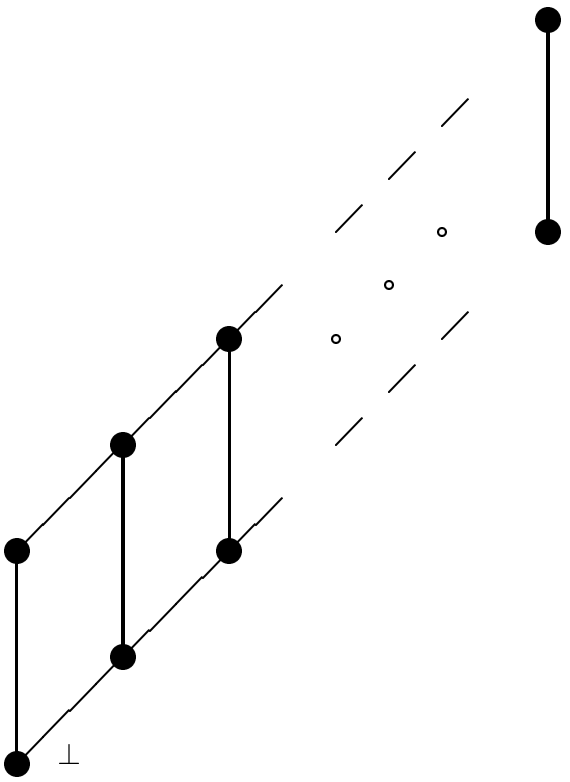


Fig. 2. The infinite ladder: a unique element covered by the top

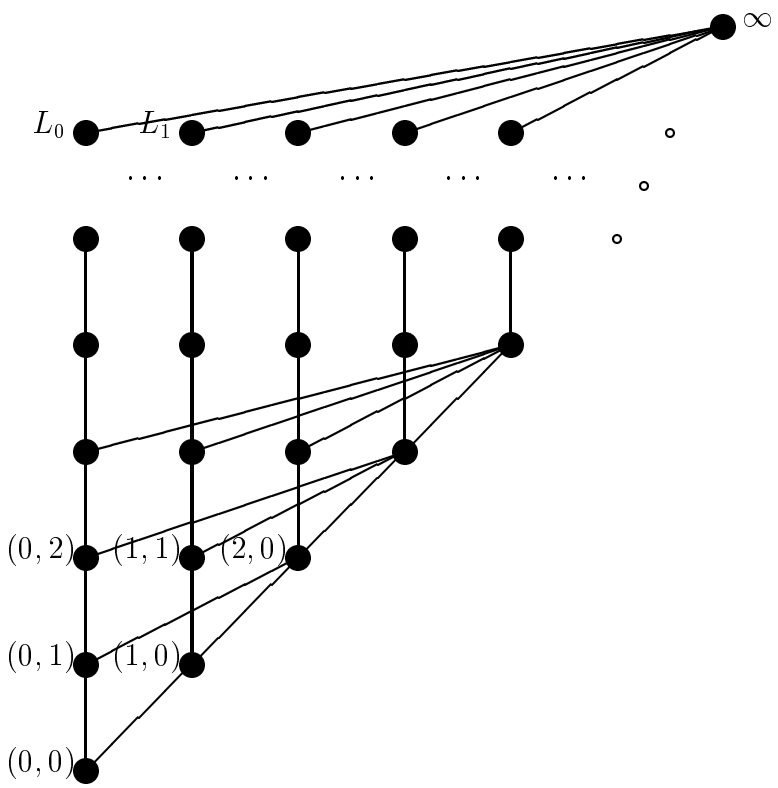


Fig. 3. A non-isolated element whose lower cover is complete

Scott domain, there is a directed set Y of isolated elements such that $x = \bigsqcup Y$. Let $< x >:= \{z_1, z_2, \cdots, z_m\}$. Clearly, not every member of Y is dominated by z_1, which means $y_1 \not\sqsubseteq z_1$ for some $y_1 \in Y$. Similarly, for each $2 \leq i \leq m$, there is some $y_i \in Y$ such that $y_i \not\sqsubseteq z_i$. Because Y is directed, and $< x >$ is finite, there is some $y^* \in Y$ which is not dominated by any of the z_i $(1 \leq i \leq m)$: just take y^* to be any element in Y which dominates all z_i, $1 \leq i \leq m$. In fact, this y^* must be equal to x since we have no other choice: the completeness of $< x >$ guarantees that if y^* is strictly below x it must be below one of z_is. Therefore, x is isolated, since y^* is.

□

Note that the condition "$< x >$ is finite" cannot be dropped. An example to illustrate this is given as follows (see Figure 3). Let $(D, \sqsubseteq)$ be the partial order where

$$D = \{(i,j) \mid i,j \in \omega \ \& \ j \geq i\} \cup \{L_k \mid k \in \omega\} \cup \{\infty\},$$

such that $(i,j) \sqsubseteq (i',j')$ iff either $i = i'$ and $j \leq j'$, or $j < i'$, and for each $i \in \omega$, $L_i = \bigsqcup_{j \in \omega}(i,j)$, and $\infty = \bigsqcup_{i \in \omega}(i,i)$. It can be shown that $\{L_i \mid i \in \omega\}$ is a complete set of lower cover of ∞, but ∞ is not an isolated element.

3 Complete Primes and Quasi-Primes

We now proceed to compare quasi-primes with complete primes. This will make clear why we choose the name "quasi-primes". We recall some definitions first.

Definition 5 *An element x of a dcpo D is a complete prime if for each set X (including the empty set),*

$$x \sqsubseteq \bigsqcup X \Rightarrow \exists y \in X.\ x \sqsubseteq y.$$

A Scott domain D is prime algebraic if every element of D is the join of the set of complete primes below it.

This definition does not immediately show any apparent connection between complete primes and quasi-primes. However, our next theorem reveals that a quasi-prime has a similar, but weaker, property as that of complete primes: for each X,

$$q = \bigsqcup X \Rightarrow \exists a \in X.\ q = a.$$

Note the only difference with complete primes is that we replaced the the $\sqsubseteq$'s by $=$'s. It turns out that this characterization of quasi-primes gives exactly the *completely join-irreducible elements*, as in [5].

As pointed out in the previous section, simply because $< q >$ is a singleton it does not mean that q is a quasi-prime. We require that, in addition, q should be isolated. There is another point we have to be careful about. Our definition of quasi-primes did not say anything about the completeness of $< q >$. Even if $< q >$ is a singleton, we cannot automatically equate this with the property

$$\exists q^{\prec} \sqsubset q.\ (x \sqsubset q \Rightarrow x \sqsubseteq q^{\prec}),$$

i.e., every element strictly below q must be below the member in $< q >$. We need $< q >$ to be complete to make such a conclusion. However, the following lemma shows that it is safe for us to assume the completeness of $< q >$, provided that q is isolated.

Lemma 6 *Let d be an isolated element of a dcpo D. Then $< d >$ is complete.*

PROOF. We use Zorn's lemma. Let $x \sqsubset d$. Consider the subset $[x, d)$, where

$$[x, d) := \{y \mid x \sqsubseteq y \sqsubset d\}.$$

With the order of D restricted to this set, we get a partial order again. Let L be any chain form $[x, d)$. Because L is directed, $\bigsqcup L$ exists in D. Clearly $\bigsqcup L \sqsubseteq d$. The equality cannot occur since d is isolated. Therefore L has an

upper bound in $[x, d)$. By Zorn's lemma, $[x, d)$ has a maximal element. It is easy to see that this maximal element belongs to $< d >$, and it is above x. Therefore, $< d >$ is complete.

□

We are now ready to prove the following theorem. Note that we do not have to assume D to be a Scott domain.

Theorem 7 *In a bounded complete dcpo D, the following are equivalent:*

(i) *q is a quasi-prime,*
(ii) *for every X, $q = \bigsqcup X$ implies $q \in X$, and*
(iii) *there exists a unique $q^{\prec} \sqsubset q$ such that $y \sqsubset q$ implies $y \sqsubseteq q^{\prec}$.*

PROOF. $(i) \Rightarrow (ii)$: This is because, by the previous lemma, $< q >$ is unique and complete.

$(ii) \Rightarrow (iii)$: Clearly $q \neq \bot$. Let $[\bot, q)$ be the set of members strictly below q, *i.e.*,

$$[\bot, q) := \{d \mid \bot \sqsubseteq d \sqsubset q\}.$$

This set must be directed (note that it is not empty). For any two members d_1, d_2 in $[\bot, q)$, we clearly have $d_1 \sqcup d_2 \sqsubseteq q$. But $d_1 \sqcup d_2 = q$ cannot happen because of 2 and the fact that d_1, d_2 are both strictly below q. Therefore $d_1 \sqcup d_2 \in [\bot, q)$ and $[\bot, q)$ is directed. It is now an easy step to check that

$$< q >= \{\bigsqcup[\bot, q)\}$$

and we can just take $q^{\prec}$ to be $\bigsqcup[\bot, q)$.

$(iii) \Rightarrow (i)$: It is easy to see that $< q >= \{q^{\prec}\}$, where $q^{\prec}$ is the unique element with the property stated in 3. We only need to show that q is isolated. However, $< q >$ is complete. By Lemma 4, it must be isolated.

□

We do not even have to assume that D is bounded complete in the previous theorem. With almost the same proof, it can be shown that the following is true.

Theorem 8 *In any dcpo D, the following are equivalent:*

(i) *q is a quasi-prime,*
(ii) *for every X, $q \in \mathsf{mub}(X)$ implies $q \in X$, and*
(iii) *there exists a unique $q^{\prec} \sqsubset q$ such that $y \sqsubset q$ implies $y \sqsubseteq q^{\prec}$.*

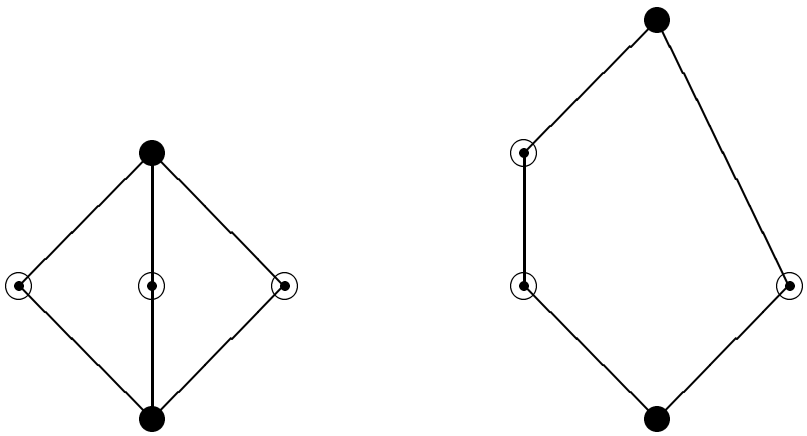


Fig. 4. Two quasi-prime algebraic domains which are not prime algebraic

Here $\mathsf{mub}(X)$ *stands for the set of minimal upper bounds of* X*.*

There are two useful corollaries of Theorem 7.

Corollary 9 *In a Scott domain, every complete prime is a quasi-prime. As a consequence, we have*

(i) every complete prime is an isolated element with a unique element immediately below it, and

(ii) if p *is a complete prime then* $\exists p^{\prec} \sqsubset p.\ (y \sqsubset p \Rightarrow y \sqsubseteq p^{\prec})$.

Corollary 10 *Prime algebraic domains are quasi-prime algebraic.*

There are, however, quasi-prime algebraic domains which are not prime algebraic. Figure 3 contains two finite domains which are quasi-prime algebraic (by Theorem 2) but not prime algebraic. The circled elements are quasi-primes.

We end this section with a result relating prime algebraic domains to distributivity.

Theorem 11 *Let* D *be a quasi-prime algebraic domain. It is prime algebraic if and only if it satisfies the infinite distributive law:*

$$x \sqcap \bigsqcup X = \bigsqcup \{x \sqcap a \mid a \in X\}.$$

PROOF. Only if: The proof goes by representing any element by the set of complete primes below it. The infinite distributive law then follows from the corresponding laws for sets.

If: Let D be a quasi-prime algebraic domain satisfying the infinite distributive law. It is enough to show that every quasi-prime q must actually be a complete prime. Let $q \sqsubseteq \bigsqcup X$. By the infinite distributive law, we have

$$q \;=\; q \sqcap \bigsqcup X = \bigsqcup\{q \sqcap a \mid a \in X\}.$$

By Theorem 7, this implies $q = q \sqcap a$ for some $a \in X$. Therefore, $q \sqsubseteq a$ for some $a \in X$.

□

4 Linear Functions and Quasi-Linear Functions

Let $f : D \to E$ be a continuous function, where D and E are algebraic. If $e \sqsubseteq f(x)$ for some isolated element $e \in E$, then there is an isolated element $d \sqsubseteq x$ in D such that $e \sqsubseteq f(d)$. This follows from algebraicity: by representing x as the least upper bounds of a directed set X of isolated elements, the continuity of f ensures that $e \sqsubseteq \bigsqcup\{f(y) \mid y \in X\}$, with $\{f(y) \mid y \in X\}$ again directed.

If, on the other hand, f has the property that whenever $e \sqsubseteq f(x)$ for some isolated element $e \in E$, there is an isolated element $d \in D$ such that $e \sqsubseteq f(x)$, then f must be continuous. This again follows from algebraicity: given any directed set $X \subseteq D$, we have

$$f(\bigsqcup X) = \bigsqcup\{f(y) \mid y \in X\}$$

because given any isolated element e below $f(\bigsqcup X)$, there is some isolated element $d \sqsubseteq \bigsqcup X$ such that $e \sqsubseteq f(d)$. This implies $e \sqsubseteq f(y)$ for some $y \in X$, and hence, $e \sqsubseteq \bigsqcup\{f(y) \mid y \in X\}$.

Therefore, we have shown that a function f on algebraic domains is continuous if and only if

$$e \sqsubseteq f(x) \Rightarrow \exists d \sqsubseteq x.\, e \sqsubseteq f(d),$$

where d, e are isolated. By an extremely similar proof, one can show that

Proposition 12 *A function f on prime algebraic domains is linear if and only if*

$$q \sqsubseteq f(x) \Rightarrow \exists p \sqsubseteq x.\, q \sqsubseteq f(p),$$

where p, q are complete primes.

What is the corresponding notion of morphisms on quasi-prime algebraic domains? We take a similar property as the one for linear functions mentioned above *as the definition* for a quasi-linear function.

Definition 13 *Let D, E be quasi-prime algebraic domains and D^q, E^q quasi-primes of D and E, respectively. A function $f : D \to E$ is quasi-linear if*

$$\forall x \in D\ \forall e \in E^q.\ [e \sqsubseteq f(x)\ \Rightarrow\ \exists d \in D^q.\ d \sqsubseteq x\ \&\ e \sqsubseteq f(d)].$$

The following proposition provides some basic properties about quasi-linear functions. Although these are simple properties which may not be hard to verify, they are essential for us: without these properties the notion of a quasi-linear function would not make much sense.

Proposition 14 *Quasi-linear functions have the following properties:*

(i) every identity is quasi-linear,
(ii) composition preserves quasi-linear functions, and
(iii) quasi-linear functions are continuous.

PROOF. (i), (ii) are straightforward. We give a proof for (iii) here.

Let $f : D \to E$ be a quasi-linear function and let X be a directed set of D. It is enough to show that

$$f(\bigsqcup X) \sqsubseteq \bigsqcup\{f(y) \mid y \in X\}.$$

The quasi-prime algebraicity of D and E makes it sufficient to show that every quasi-prime below $f(\bigsqcup X)$ is also below $\bigsqcup\{f(y) \mid y \in X\}$. Let $q \sqsubseteq f(\bigsqcup X)$, where q is a quasi-prime. Because f is quasi-linear, there is a quasi-prime $p \sqsubseteq \bigsqcup X$ such that $q \sqsubseteq f(p)$. However, p is isolated. Therefore, $p \sqsubseteq x$ for some x in X, which in turn implies $q \sqsubseteq f(x)$ for some x in X. Hence,

$$q \sqsubseteq \bigsqcup\{f(y) \mid y \in X\}.$$

□

Quasi-linear functions are, however, not necessarily linear. To see this, we recall the remark made in the introduction: a non-trivial step function $[a, b]$ is linear if and only if a is a complete prime. The following result shows that it is as easy to find a quasi-linear, but non-linear function as to find a quasi-prime which is not a complete prime: just take a to be a quasi-prime but not a complete prime.

Proposition 15 *Let D, E be quasi-prime algebraic domains. A non-trivial step function $[a, b]$ is quasi-linear if and only if a is a quasi-prime.*

PROOF. If: Suppose a is a quasi-prime and $q \sqsubseteq [a, b](x)$. The bottom is never a quasi-prime; therefore, $a \sqsubseteq x$. We clearly have $q \sqsubseteq [a, b](a)$.

Only if: Suppose $[a, b]$ is a non-trivial, quasi-linear function. We have $b \neq \perp_E$. By the quasi-prime algebraicity of E, there exists a quasi-prime q below b. Thus, $q \sqsubseteq [a, b](a)$. This implies $q \sqsubseteq [a, b](p)$ for some $p \sqsubseteq a$, which is only possible if $a = p$. Therefore, a is a quasi-prime.

□

The next result, whose easy proof is omitted, shows that quasi-linear functions are indeed a robust generalization of linear functions.

Proposition 16 *A function between prime algebraic domains is linear if and only if it is quasi-linear.*

5 Topological Aspects

This section discusses the topological aspects of quasi-linear functions.

Recall that a topology on a set S is a collection of subsets of S that is closed under finite intersection and arbitrary union. A set S and a topology $\Re$ on it form a topological space $(S, \Re)$. A *base* of the topology $\Re$ on S is a subset $\Re_1 \subseteq \Re$ such that every open set is the union of elements of $\Re_1$. Let $(D, \sqsubseteq)$ be a cpo. A subset O of D is said to be *Scott open* if O is upwards closed, *i.e.*, for any x in O, $x \sqsubseteq y$ implies y in O, and whenever $X \subseteq D$ is directed and $\bigsqcup X$ is in O, we have $X \cap O \neq \emptyset$. The *Scott topology* on a cpo $(D, \sqsubseteq)$ consists of all the Scott open sets of $(D, \sqsubseteq)$, written $\Omega(D)$.

Scott topology characterizes continuous functions. A function f from D to E is continuous if and only if the inverse image of any Scott open set is Scott open. Let $f, g : D \to E$ be continuous functions. $f \sqsubseteq g$ if and only if for every Scott open O of E, $f^{-1}(O) \subseteq g^{-1}(O)$.

Proposition 17 $\uparrow x$ *is open if and only if* $x \in D^0$, *where*

$$\uparrow x = \{ y \in D \mid x \sqsubseteq y \}.$$

Open sets of this form are called *prime open* since they are the complete primes in the lattice $(\Omega(D), \subseteq)$.

Proposition 18 *The set* $\{ \uparrow x \mid x \in D^0 \}$ *is a base for the Scott topology of an* ω*-algebraic domain* D.

So, every Scott open set is a union of open sets of the form $\uparrow x$ with $x \in D^0$. The empty set, being open, is the union of the empty collection of prime open sets.

Is there a topology which characterizes quasi-linear functions? We know that there is no topology which characterizes stable functions [20]. For a similar reason, there is no topology which characterizes either linear functions or quasi-linear functions. However, short of being closed under finite intersection, there are collections of Scott open sets which characterizes quasi-linear functions and linear functions.

Definition 19 *Let D be a Scott domain. A Scott open set O is called prime generated if there is a set X of complete primes of D such that*

$$O = \{y \mid \exists x \in X.\ y \sqsupseteq x\}.$$

Similarly, a Scott open set O is called quasi-prime generated if there is a set X of quasi-primes of D such that

$$O = \{y \mid \exists x \in X.\ y \sqsupseteq x\}.$$

We have the following theorem, which seems to be the best one can get concerning the topological characterization of linear and quasi-linear functions. We need a lemma first, whose proof is straightforward.

Lemma 20 *Let D be a quasi-prime algebraic domain. An open set O is quasi-prime generated if and only if for each y in O, there is a quasi-prime q below y, such that q is also in O.*

Theorem 21

(i) Let D, E be prime algebraic domains. A function $f : D \to E$ is linear if and only if the inverse image of f preserves prime generated open sets.
(ii) Let D, E be quasi-prime algebraic domains. A function $f : D \to E$ is quasi-linear if and only if the inverse image of f preserves quasi-prime generated open sets.

PROOF. As an example, we show the second conclusion.

Only if: Let $f : D \to E$ be a quasi-linear function between quasi-prime algebraic domains. Given any quasi-prime generated open set O, there is a set of quasi-primes $X \subseteq E^q$ such that

$$O = \{y \mid \exists x \in X.\ y \sqsupseteq x\},$$

by definition. We show that $f^{-1}(O)$ is again quasi-prime generated. Let d be a member of $f^{-1}(O)$, or $f(d)$ a member in O. We have, $x \sqsubseteq f(d)$ for some x in X. By Definition 13, there is a quasi-prime $q \sqsubseteq d$ such that $x \sqsubseteq f(q)$. Hence, q belongs to $f^{-1}(O)$. By the previous lemma, $f^{-1}(O)$ is quasi-prime generated.

If: Use Definition 13 again.

□

Part II. Categories

This part studies categories of quasi-prime algebraic domains. After briefly presenting a linear category of prime algebraic lattices, we introduce a monoidal closed category of quasi-prime algebraic domains with quasi-linear functions as morphisms. We also introduce a cartesian closed category of quasi-prime algebraic domains with Scott continuous functions. All our results in this section are shown with the technical tool of information systems in the sense of Scott [14].

6 Prime Algebraic Lattices: A Linear Category

This section reviews a linear category of prime algebraic lattices. We use information systems because they serve as a bridge to the sections to follow.

We restrict ourselves to consistently complete, ω-algebraic dcpos in the rest of this paper. This means whenever algebraicity is concerned, a countable number of isolated elements is implicitly implied.

Prime algebraic domains have been discussed by a number of people [7,8,21,18,17] when domain theoretic models of linear logic were studied. In particular, [21] contains an outline of a monoidal closed category of prime algebraic domains with non-stable, linear functions which is further elaborated here. In this section we show that within prime algebraic domains, the class of prime algebraic lattices is even better: they give rise to a linear category. Although the essence of these results can be derived from [7,8,21,18,17] without much difficulty, we believe our representation using prime information systems offers a pleasing easy access to these results.

Definition 22 *A prime information system is a structure*

$$\underline{A} = (A, \vdash)$$

where A is a countable set of tokens (propositions), and $\vdash$ is a preorder on A.

These are special kind of information systems. Originally, prime information systems were introduced by Zhang in [18] for a representation of dI-domains. Apart from the fact that we do not consider the stable order here, some other simplifications are made:

– We do not have a consistency predicate, thus everything is consistent;
– We do not require an axiom similar to Axiom I for dI-domains;

– Further more, $\vdash$ here is a preorder ($a \vdash b$), rather than a relation between finite sets of tokens and tokens ($X \vdash a$).

The preorder $\vdash$ is intuitively understood as the entailment (that is why $\vdash$ is required to be a preorder). When $a \vdash b$ we say a entails b.

Definition 23 *Given any subset u of tokens, the set*

$$\overline{u} := \{b \mid \exists a \in u.\, a \vdash b\}$$

is called the deductive closure of u. A set x is deductively closed if $x = \overline{x}$. Write $\mid \underline{A} \mid$ for the collection of deductively closed subsets of $\underline{A}$, called its elements.

Note that $\overline{u} = \overline{\overline{u}}$. Note also that ordered under inclusion, $\mid \underline{A} \mid$ is a partial order. Join in this partial order is set union, *i.e.*, for any subset X of elements of $\underline{A}$,

$$\bigsqcup X = \bigcup X.$$

Clearly, A itself is an ideal element. The empty set is deductively closed, and it is the least element. For each single token a, $\overline{a}$ is deductively closed (for singleton sets we omit the brackets). If $\overline{a} \subseteq \bigsqcup X$, where X is a set of elements, then $a \in \bigcup X$. This implies $b \vdash a$ for some $b \in \bigcup X$. Thus $\overline{a} \subseteq x$ for some $x \in X$. Moreover, it is routine to check that, for each element x, $x = \bigsqcup\{\overline{a} \mid a \in x\}$. Thus we have the following theorem.

Theorem 24 *For a prime information system $\underline{A}$, $(\mid \underline{A} \mid, \subseteq)$ is a prime algebraic lattice. The complete primes are exactly of the form $\overline{a}$, with $a \in A$.*

On the other hand, each prime algebraic lattice can be associated with a prime information system. In this sense prime information systems are a representation of prime algebraic lattices.

Theorem 25 *Let D be a prime algebraic lattice. Then there is a prime information system $\mathcal{I}(D)$ such that $(D, \sqsubseteq)$ is isomorphic to $(\mid \mathcal{I}(D) \mid, \subseteq)$.*

PROOF. Given a prime algebraic lattice D, it is easy to check that

$$\mathcal{I}(D) := (D^p, \vdash)$$

is a prime information system, where D^p is the set of complete primes of D, and $a \vdash b$ iff $b \sqsubseteq a$. For each $d \in D$, let

$$d \longmapsto \{e \mid e \in D^p \;\&\; e \sqsubseteq d\}.$$

This is one-one and onto, because of prime algebraicity. Moreover, $d_1 \sqsubseteq d_2$ if and only if the set of complete primes below d_1 is included in the set of

complete primes below d_2.

□

Let D be a prime algebraic lattice. Its dual D^{op} is again a lattice. But is it also prime algebraic? The answer is "yes" but the proof is non-trivial: a complete prime of D may not be a complete prime of D^{op}, which can be seen from Theorem 7. The diamond with a new top is a concrete example.

However, from the point of a prime information system $(A, \vdash)$ representing D, this becomes intuitively straightforward, because we can just take the dual of the preorder $\vdash$ as a candidate prime information system for D^{op}. Indeed, we have the following result, whose proof takes advantage of the concrete structure of a prime information system.

Theorem 26 *Let $(D, \sqsubseteq)$ be a prime algebraic lattice. Then*

$$D^{\mathrm{op}} \cong (\mid \mathcal{I}(D)^{\mathrm{op}} \mid, \subseteq),$$

where $\mathcal{I}(D)$ is the prime information system defined in the proof of the previous theorem, and $\mathcal{I}(D)^{\mathrm{op}}$ is the dual of the preorder $\mathcal{I}(D)$.

PROOF. Let $(D, \sqsubseteq)$ be a prime algebraic lattice. Let f be the mapping from D to $\mid \mathcal{I}(D)^{\mathrm{op}} \mid$ such that

$$f(d) = \{t \mid t \in D^p \;\&\; t \not\sqsubseteq d\}.$$

For each $d \in D$, $\{t \mid t \in D^p \;\&\; t \not\sqsubseteq d\}$ is an element of $\mathcal{I}(D)^{\mathrm{op}}$. This because if

$$q \in \{t \mid t \in D^p \;\&\; t \not\sqsubseteq d\}$$

and $q \vdash r$ in $\mathcal{I}(D)^{\mathrm{op}}$, then r is a complete prime of D with $q \sqsubseteq r$. Hence,

$$r \in \{t \mid t \in D^p \;\&\; t \not\sqsubseteq d\}.$$

This means f is well-defined.

The function f is clearly one-to-one. We check that it is onto. Let x be any ideal element of $\mathcal{I}(D)^{\mathrm{op}}$. Consider the element $\bigsqcup(D^p - x)$, *i.e.*, the least upper bound of complete primes of D not belonging to x. We claim that

$$x = f(\bigsqcup D^p - x) = \{t \mid t \in D^p \;\&\; t \not\sqsubseteq \bigsqcup(D^p - x)\}.$$

This claim clearly holds if x is empty. Suppose $q \in x$, where q is a complete prime in D^p. If $q \sqsubseteq \bigsqcup(D^p - x)$, then $q \sqsubseteq r$ for some $r \in D^p - x$. But $q \in x$

and $q \sqsubseteq r$ implies $r \in x$, which is a contradiction. Therefore, we must have $q \not\sqsubseteq \bigsqcup(D^p - x)$, which implies $q \in f(\bigsqcup D^p - x)$.

Suppose, on the other hand, $q \in f(\bigsqcup D^p - x)$. We have $q \not\sqsubseteq \bigsqcup(D^p - x)$, and q is a complete prime. Clearly this means $q \notin D^p - x$, so q must be in x.

Now observe that for $d, e \in D$, $d \sqsubseteq e$ if and only if

$$\{t \mid t \in D^p \;\&\; t \not\sqsubseteq d\} \supseteq \{t \mid t \in D^p \;\&\; t \not\sqsubseteq e\}.$$

Therefore, f gives the desired isomorphism between D^{op} and $(\mid \mathcal{I}(D)^{\mathrm{op}} \mid, \subseteq)$.

□

To get a category of prime information systems, we have to specify the morphisms. This is based on Scott's idea of an approximable mapping.

Definition 27 *An approximable mapping from a prime information system $(A, \vdash_A)$ to a prime information system $(B, \vdash_B)$ is a relation $R \subseteq \mathrm{Fin}(A) \times B$ such that*

$$[X \vdash_A X' \;\&\; (X', b') \in R \;\&\; b' \vdash_B b] \Rightarrow (X, b) \in R.$$

Here $\mathrm{Fin}(A)$ is the collection of finite subsets of A, and $X \vdash X'$ is the abbreviation for

$$\forall a' \in X' \exists a \in X.\; a \vdash a'.$$

An approximable mapping is linear if it is a relation $R \subseteq A \times B$ such that

$$[a \vdash_A a' \;\&\; (a', b') \in R \;\&\; b' \vdash_B b] \Rightarrow (a, b) \in R.$$

It is important to note that the abbreviation $X \vdash_A X'$ here can be seen as a specialization of one for general information systems (see next section), in the sense that $X \vdash_A X'$ stands for $X \vdash a'$ for each a' in X'. This is because in the general case, $X \vdash a'$ does not imply the existence of an a in X such that $a \vdash a'$, although it is always assumed that if $a \vdash a'$ and $a \in X$, then $X \vdash a'$, for consistent X. The following results show that approximable mappings are closely related to continuous functions.

Theorem 28 *Suppose $R : \underline{A} \to \underline{B}$ is an approximable mapping. Let $\mathcal{F}(R)$ be specified by*

$$\mathcal{F}(R)(x) = \{\, b \mid \exists X \subseteq x.\, X \,R\, b \,\}.$$

Then $\mathcal{F}(R)$ is a continuous function from $\mid \underline{A} \mid$ to $\mid \underline{B} \mid$. Moreover, $\mathcal{F}(R)$ is linear if R is linear.

PROOF. First we check that $\mathcal{F}(R)$ is well defined. We show that given any element x of $\underline{A}$, $\mathcal{F}(R)(x)$ is closed under entailment. Assume $a \in \mathcal{F}(R)(x)$

and $a \vdash_B b$. This means $X\,R\,a$ for some $X \subseteq x$. By the definition of an approximable mapping, we have $X\,R\,b$, and therefore, $b \in \mathcal{F}(R)(x)$.

The proof for the continuity of $\mathcal{F}(R)$ is quite standard, based on the finiteness of the first component of R.

Let R be a linear approximable mapping and let

$$b \in \mathcal{F}\,R\,(\bigcup_{i\in I} x_i).$$

Because R is linear, there is some $a \in \bigcup_{i\in I} x_i$ such that $a\,R\,b$. This implies $a \in x_i$ for some $i \in I$ and hence,

$$b \in \bigcup\{\mathcal{F}\,R\,(x_i) \mid i \in I\}.$$

We have shown

$$\mathcal{F}\,R\,(\bigcup_{i\in I} x_i) \subseteq \bigcup\{\mathcal{F}\,R\,(x_i) \mid i \in I\},$$

which is sufficient for $\mathcal{F}(R)$ to be linear.

□

One can show further that every linear function is determined by a linear approximable map. However, that is a consequence of Theorem 29 below, so we omit the proof.

Prime information systems with approximable mappings give rise to a category. The identity morphism is the approximable mapping

$$\{(X, b) \mid \exists a \in X. a \vdash b\}.$$

The composition of approximable mappings R and S is the relational composition of $\check{}R$ with S, where

$$\check{}R := \{(X, Y) \mid \forall b \in Y.\ (X, b) \in R\}.$$

Similarly, prime information systems with linear approximable mappings constitutes a category. We write **PIS.a** and **PIS.l** for the categories of prime information systems with approximable mappings and prime information systems with linear approximable mappings, respectively.

Theorem 29 **PIS.a** *is equivalent to* **PAL.c** *and* **PIS.l** *is equivalent to* **PAL.l**. *Here* **PAL.c** *and* **PAL.l** *are categories of prime algebraic lattices with continuous functions and prime algebraic lattices with linear functions, respectively.*

PROOF. As an example, we show that **PIS.l** is equivalent to **PAL.l**. By a result of MacLane [12], it is enough to show that there is full and faithful

functor from **PIS.l** to **PAL.l**, and each prime algebraic lattice D is isomorphic to $(|\ \underline{A}\ |, \subseteq)$ for some prime information system $\underline{A}$. The latter is the content of Theorem 25, so it remains to show that there is a full and faithful functor from **PIS.l** to **PAL.l**.

The desired functor $\mathcal{F}$ maps each prime information system $\underline{A}$ to the prime algebraic lattice $(|\ \underline{A}\ |, \subseteq)$. It maps each linear approximable mapping R to the linear function

$$\mathcal{F}(R) := \lambda x.\{b \mid \exists a \in x.(a,b) \in R\}.$$

$\mathcal{F}$ is a functor because it preserves identities and composition.

First we show that $\mathcal{F}$ is full. Suppose $\underline{A}$ and $\underline{B}$ are prime information systems and

$$f :|\ \underline{A}\ | \rightarrow |\ \underline{B}\ |$$

a linear function. Define a relation $R \subseteq A \times B$ by letting $(a,b) \in R$ if $b \in f(\overline{a})$. R is clearly a linear approximable mapping. Moreover, the linear function $\mathcal{F}(R)$ determined by R is actually equal to f. To see this, let x be an element of $\underline{A}$. We have

$$\begin{aligned} b \in \mathcal{F}(R)(x) &\Leftrightarrow \exists a \in x\ (a,b) \in R \\ &\Leftrightarrow \exists a \in x\ b \in f(\overline{a}) \\ &\Leftrightarrow b \in f(x)\ (\text{ Since f is linear}). \end{aligned}$$

Suppose $R,\ S :\ \underline{A} \rightarrow \underline{B}$ are linear approximable mappings such that $\mathcal{F}R$ equals $\mathcal{F}S$. To show $R = S$, let (a,b) be a member of R. We have

$$\begin{aligned} (a,b) \in R &\Rightarrow b \in \mathcal{F}(R)(\overline{a}) \\ &\Rightarrow b \in \mathcal{F}(S)(\overline{a}) \\ &\Rightarrow \exists a' \in \overline{a}\ (a',b) \in S \\ &\Rightarrow \exists a'\ a \vdash_A a'\ \&\ (a',b) \in S \\ &\Rightarrow (a,b) \in S. \end{aligned}$$

By symmetry, we have $S \subseteq R$. Therefore, $S = R$ and $\mathcal{F}$ is faithful.

□

Let $\underline{A} = (A, \vdash_A)$, $\underline{B} = (B, \vdash_B)$ be prime information systems. The following

constructions can be introduced on prime information systems.

$$\underline{A} \otimes \underline{B} = (A \times B, \vdash_{A\otimes B}), \quad \text{where } (a,b) \vdash_{A\otimes B} (a',b') \text{ iff } a \vdash_A a' \ \& \ b \vdash_B b';$$
$$\underline{A} \multimap \underline{B} = (A \times B, \vdash_{A\multimap B}), \quad \text{where } (a,b) \vdash_{A\multimap B} (a',b') \text{ iff } a' \vdash_A a \ \& \ b \vdash_B b';$$
$$\underline{A} \times \underline{B} = (\{0\} \times A \cup \{1\} \times B, \vdash_{A\times B}), \quad \text{where } (i,a) \vdash_{A\times B} (j,a') \text{ iff}$$
$$(i = j = 0 \ \& \ a \vdash_A a') \ \text{ or } (i = j = 1 \ \& \ a \vdash_B a');$$
$$\bot = (\{\bullet\}, \{(\bullet,\bullet)\}).$$

We follow the notation and terminology introduced in [1,15] except $\bot$, $\top$, 0, and 1 for which we follow Girard [6]. A linear category is a symmetric monoidal closed category with all finite products and a dualizing object.

Theorem 30 **PIS.l** *is a linear category.* **PIS.a** *is a cartesian closed category.*

PROOF. We only give a proof of the first conclusion here. The prime information system $(\emptyset,\emptyset)$ is both initial and terminal. The binary product is $\times$, and the tensor product is $\otimes$. One can check that $\otimes : \mathbf{PIS.l} \times \mathbf{PIS.l} \to \mathbf{PIS.l}$ is a functor satisfying the MacLane-Kelly coherence conditions [12], where the tensor unit is $\bot$. We also have

$$\underline{A} \otimes \underline{B} \multimap \underline{C} \cong \underline{A} \multimap [\underline{B} \multimap \underline{C}].$$

Since the linear approximable mappings from $\underline{A}$ to $\underline{B}$ is in one-one correspondence with the elements of $\underline{A} \multimap \underline{B}$, the above isomorphism is a key fact for monoidal closure.

PIS.l has a dualizing object $\bot$. It is apparent that $\underline{A} \cong [[\underline{A} \multimap \bot] \multimap \bot]$. Other operators can be defined via the dualizing object. For coproduct $+$, we have

$$\underline{A} + \underline{B} := (\underline{A}^\bot \times \underline{B}^\bot)^\bot \cong \underline{A} \times \underline{B};$$

For tensorsum $\odot$, we have

$$\underline{A} \odot \underline{B} := (\underline{A}^\bot \otimes \underline{B}^\bot)^\bot \cong \underline{A} \otimes \underline{B}.$$

□

Note that gluing the bottoms and the tops of two prime algebraic domains does not give back a prime algebraic domain.

PIS.l is, moreover, a Girard category [15]. Let

$$\begin{aligned}
\underline{A} \to \underline{B} &= (\mathrm{Fin}(A) \times B, \vdash_{A\to B}), \quad \text{with} \\
&\quad (u,a) \vdash_{A\to B} (v,b) \ \text{ iff } v \vdash_A u \ \& \ a \vdash_B b; \\
!\underline{A} &= (\mathrm{Fin}(A), \vdash_{!A}) \ \text{ with} \\
&\quad u \vdash_{!A} v \ \text{ iff } u \vdash_A v,
\end{aligned}$$

where $\mathrm{Fin}(A)$ stands for the set of finite subsets of A.

The function space $\to$ can be decomposed into two more primitive constructions mentioned above. This is the content of the following theorem.

Theorem 31 *Let $\underline{A}, \underline{B}$ be prime algebraic information systems. Then*

$$\underline{A} \to \underline{B} \cong !\underline{A} \multimap \underline{B}.$$

PROOF. We have

$$\begin{aligned}
(u,a) \vdash_{A\to B} (v,b) &\Leftrightarrow v \vdash_A u \ \& \ a \vdash_B b \\
&\Leftrightarrow v \vdash_{!A} u \ \& \ a \vdash_B b \\
&\Leftrightarrow (u,a) \vdash_{!A \multimap B} (v,b).
\end{aligned}$$

□

7 Irreducible Information Systems

Instead of directly working with quasi-prime algebraic domains, we introduce a representation of them as *irreducible information systems*. As is explained in detail below, the term "irreducible" refers to an information system without useless tokens with respect to the underlying domain it represents. In this way, the abstract notion of a partial order will be replaced by set inclusion in a certain space. This will make the presentation and proof of the results in the next section much neater, just as we have seen for the case of prime algebraic lattices.

Definition 32 *An information system is a structure* $\underline{A} = (\, A,\ Con,\ \vdash\,)$ *where*

- A *is a countable set of propositions (tokens),*
- Con *is a collection of finite subsets of* A*, called the consistent sets,*
- $\vdash \subseteq Con - \{\emptyset\} \times A$, *the entailment relation,*

which satisfy

- $(X \subseteq Y \ \&\ Y \in Con) \Rightarrow X \in Con$,
- $a \in A \Rightarrow \{\, a \,\} \in Con$,
- $(X \vdash a \ \&\ X \in Con) \Rightarrow X \cup \{\, a \,\} \in Con$,
- $(a \in X \ \&\ X \in Con) \Rightarrow X \vdash a$,
- $(X \vdash Y \ \&\ Y \vdash c\,) \Rightarrow X \vdash c$.

Here $X \vdash Y$ is the abbreviation for $X \vdash b$ for every $b \in Y$ and $X \dashv\vdash Y$ will be used as the abbreviation for $X \vdash Y$ and $Y \vdash X$. Note that the information systems we consider here are slightly different from those introduced by Scott. We do not assume a distinguished element Δ, standing for *true*. Also, we require that X is non-empty when we write $X \vdash a$. This has the effect that the bottom element of the corresponding domain is always the empty set.

Information systems provide a representation of Scott domains. The connection between an information system and a Scott domain is via the notion of an ideal element (or an element).

Definition 33 *The elements* $|\, \underline{A} \,|$*, of an information system*

$$\underline{A} = (\, A,\ Con,\ \vdash\,)$$

consists of subsets x *of propositions which are*

- *consistent:* $X \subseteq^{fin} x \Rightarrow X \in Con$, *and*
- *deductively closed:* $X \subseteq x \ \&\ X \vdash a \Rightarrow a \in x$.

We want to find a class of information systems which represent exactly the quasi-prime algebraic domains. It is interesting to see that this has to do with the existence of the most efficient way to represent a domain.

Definition 34 *Let* $\underline{A} = (\, A,\ Con,\ \vdash\,)$ *be an information system. A subset* B *of* A *is called redundant if for each* b *in* B*, there is a finite subset* X *of* $A - B$

such that

$$X \vdash b \;\&\; (\forall a \in X.\, b \vdash a).$$

A token is redundant if its informational content can be recaptured by a finite token set in the rest of the system. Formally, a token should be redundant exactly when the removal of it does not affect the order structure of the ideal elements. The following result confirms this expectation.

Lemma 35 *Suppose* $\underline{A} = (\, A,\ Con,\ \vdash\ )$ *is an information system and* B *is a redundant subset. If we restrict* Con *and* $\vdash$ *to the elements of* $A - B$*, we get an information system* $(A - B, Con,\ \vdash)$*, which determines an isomorphic domain as that determined by* $\underline{A}$*.*

PROOF. A routine inspection of the axioms for an information system shows that

$$\underline{A \setminus B} := (A - B,\ Con,\ \vdash)$$

is again an information system. To show that the cpos determined by $\underline{A}$ and $\underline{A \setminus B}$ are isomorphic, it is enough to establish a one-one, order preserving correspondence between the elements of the two domains. Let

$$\alpha :\ x \longmapsto x \cap (A - B)$$

and

$$\beta :\ y \longmapsto y \cup \{b \in B \mid \exists Y \subseteq y\ Y \vdash_A b\}.$$

One can verify that these mappings establishes the desired one-one correspondence.

□

Obviously, if B_1 is a redundant set of $\underline{A}$, and B_2 is a redundant set of $\underline{A \setminus B_1}$, then $B_1 \cup B_2$ is a redundant set of $\underline{A}$. However, not all information systems have the largest redundant sets, although all finite information systems do. In the finite case, redundant tokens can be removed one by one, resulting in a "minimal" information system.

Definition 36 *An information system is called irreducible if it does not contain any redundant token.*

Example 37 *Let*

$$\underline{A} = (\{1, 3\}, Con_A, \vdash_A)$$

where Con_A *includes* $\{1, 3\}$*, and* $3 \vdash_A 1$*, and let*

$$\underline{B} = (\{1, 2, 3\}, Con_B, \vdash_B)$$

where Con_B *includes* $\{1,2,3\}$*, and* $3 \vdash_B 1$, $3 \vdash_B 2$, *and* $\{1,2\} \vdash_B 3$. *Then* $\underline{A}$ *is irreducible while* $\underline{B}$ *is not, because the token* 3 *is redundant – whose informational content can be captured by* $\{1,2\}$ *already.*

Irreducible information systems are important, because they reveal an enlightening connection between a simple syntactic property of information systems and a class of domains: the quasi-prime algebraic domains.

Theorem 38 *(i) If* $\underline{A} = (\, A,\ Con,\ \vdash\,)$ *is an irreducible information system, then* $(\mid \underline{A} \mid, \subseteq)$ *is a quasi-prime algebraic domain;*

(ii) Conversely, if $(D, \sqsubseteq)$ *is a quasi-prime algebraic domain, then there exists an irreducible information system* $\underline{A}$ *such that* $(D, \sqsubseteq)$ *is isomorphic to* $(\mid \underline{A} \mid, \subseteq)$.

PROOF. (i): We first show the key fact that for an irreducible information system $\underline{A}$, each ideal element of the form $\overline{a}$, with a in A, is a quasi-prime. Note that $\overline{a} - \{a\}$ is again an ideal element, because the irreducibility of $\underline{A}$ makes it impossible for $X \vdash a$ to happen with any finite subset X of $\overline{a} - \{a\}$ – otherwise a would be redundant. Clearly, $\overline{a} - \{a\}$ is the unique element immediately below the isolated element $\overline{a}$, making $\overline{a}$ a quasi-prime. It is now easy to see that $(\mid \underline{A} \mid, \subseteq)$ is quasi-prime algebraic, for every element of $\underline{A}$ can be expressed as a union of quasi-primes.

(ii): Let D be a quasi-prime algebraic domain. Define

$$\mathcal{J}(D) := (\, \mathsf{Q}(\, D\,), Con,\ \vdash\,),$$

where $\mathsf{Q}(\, D\,)$ is the set

$$\{\, \uparrow q \mid q \text{ is a quasi-prime of } D\},$$

and $\vdash$ is a relation from the set of non-empty, finite subsets of $\mathsf{Q}(D)$ to $\mathsf{Q}(D)$ such that

$$X \vdash a \ \text{ if } \bigcap X \subseteq a, \ \text{ and } \ X \in Con \ \text{ if } \bigcap X \neq \emptyset.$$

It is easy to check that $\mathcal{J}(D)$ is an information system. We show that it is irreducible. This is because when $\bigcap_{p \in P} \uparrow p = \uparrow q$ for some finite set P of quasi-primes, we have $q = \bigsqcup P$. We can now activate Theorem 7 and arrive at the conclusion that $p = q$ for some $p \in P$. Clearly, D and $\mid \mathcal{J}(D) \mid$ are isomorphic, because both are quasi-prime algebraic, and their quasi-primes are isomorphic.

□

8 A Monoidal Closed Category with Non-Linear Functions

We now come to one of the main focuses of the paper: a monoidal closed category of quasi-linear functions. We also introduce a cartesian closed categories of quasi-prime algebraic domains. It shows that quasi-prime algebraic domains can be put in service as models of intuitionistic linear logics as well as models for typed lambda-calculi. Therefore, quasi-prime algebraic domains have properties similar to Scott domains which qualify them as a mathematical foundation for programming semantics. Once again, we follow the notations proposed by Seely [15] for various linear connectives.

Here are the four major categories to be studied in this section:

QPA.ql – quasi-prime algebraic domains with quasi-linear functions,

QPA.c – quasi-prime algebraic domains with continuous functions,

QPL.ql – quasi-prime algebraic lattices with quasi-linear functions,

QPL.c – quasi-prime algebraic lattices with continuous functions.

Correspondingly, there are four categories of irreducible information systems: **IRI.ql**, **IRI.a**, **IRL.ql**, and **IRL.a**, which we introduce now.

Definition 39 *An approximable mapping between irreducible information systems*

$$(A, Con_A, \vdash_A)$$

and

$$(B, Con_B, \vdash_B)$$

is a relation $R \subseteq Con_A \times B$ such that

$$\begin{aligned} &\forall R' \subseteq^{fin} R.\ \bigcup \pi_A(R') \in Con_A \Rightarrow \pi_B(R') \in Con_B, \\ &X \vdash_A X' \ \&\ (\forall b' \in Y\ (X', b',) \in R) \ \&\ Y \vdash_B b \\ &\qquad \Rightarrow (X, b) \in R. \end{aligned}$$

Here π_A stands for projection on A and π_B stands for projection on B. Similarly, a quasi-linear approximable mapping is a relation $R \subseteq A \times B$ such that

$$\begin{aligned} &\forall R' \subseteq^{fin} R\, \pi_A(R') \in Con_A \Rightarrow \pi_B(R') \in Con_B, \\ &Y \vdash_B b \ \&\ \forall b' \in Y \exists a' \in A(a \vdash_A a' \ \&\ (a', b',) \in R) \\ &\qquad \Rightarrow (a, b) \in R. \end{aligned}$$

Approximable mappings between conflict-free, irreducible information systems are exactly the same as above, except that the condition related to the consistency predicate can be safely dropped.

Irreducible information systems with approximable mappings form a category, denoted as **IRI.a**. Similarly, **IRI.ql** is the category of irreducible information systems with quasi-linear approximable mappings. An information system $(A, Con_A, \vdash_A)$ is *conflict-free* if every finite subset of A is a member of Con. We will simply write a conflict-free information system as $(A, \vdash_A)$, omitting the trivial consistency predicate. We write **IRL.a** for the category of conflict-free, irreducible information systems with approximable mappings, and **IRL.ql** for the category of conflict-free, irreducible information systems with quasi-linear approximable mappings.

Careful readers may want to check that indeed these are categories. This is similar to the category of prime information systems of Section 6. However, there are slight differences in the way the different kinds of approximable mappings are defined. The exact contents of the definitions are dictated by the two constraints: one is that they should correspond to continuous or quasi-linear functions, and the other is that there should be one-one correspondence between approximable mappings (quasi-linear approximable mappings) and elements of irreducible information systems for the function (quasi-linear function) space.

Theorem 40 *Categorically, we have the following:*

(i) **QPA.ql** *is equivalent to* **IRI.ql**,
(ii) **QPA.c** *is equivalent to* **IRI.a**,
(iii) **QPL.ql** *is equivalent to* **IRL.ql**, *and*
(iv) **QPL.c** *is equivalent to* **IRL.a**.

PROOF. As an example, we show that **QPL.ql** is equivalent to **IRL.ql**. A functor $\mathcal{F}$ can be given from **IRL.ql** to **QPL.ql** in the following way. $\mathcal{F}$ maps each conflict-free, irreducible information system $\underline{A}$ to $(|\ \underline{A}\ |, \subseteq)$ which, by Theorem 38, is a quasi-prime algebraic lattice. Each quasi-linear approximable mapping R determines a quasi-linear function

$$\mathcal{F}(R) := \lambda x.\{b \mid \exists a \in x.(a, b) \in R\}.$$

It is routine to show that this defines a continuous function. It is quasi-linear because for each b in $\mathcal{F}(R)(x)$, there is some a in x such that b is in $\mathcal{F}(R)(\overline{a})$. Therefore, for any quasi-prime q such that $q \subseteq \mathcal{F}(R)(x)$, there is a quasi-prime $p \subseteq x$ such that $q \subseteq \mathcal{F}(R)(p)$, as required by the definition of a quasi-linear function. Furthermore, $\mathcal{F}$ is a functor because it preserves identities and composition.

By a result of MacLane [12], it is enough to show that $\mathcal{F}$ is full and faithful, and each quasi-prime algebraic lattice D is isomorphic to $(|\ \underline{A}\ |, \subseteq)$ for some conflict-free, irreducible information system $\underline{A}$. The latter is straightforward: we can take $\underline{A}$ to be $\mathcal{J}(D)$, given in Theorem 38. It remains to show that $\mathcal{F}$ is full and faithful.

First we show that $\mathcal{F}$ is full. Suppose $\underline{A}$ and $\underline{B}$ are conflict-free, irreducible information systems and

$$f :|\ \underline{A}\ | \rightarrow |\ \underline{B}\ |$$

a quasi-linear function. Define a relation $R \subseteq A \times B$ by letting $(a, b) \in R$ if $b \in f(\overline{a})$. R is a quasi-prime linear approximable mapping. Suppose

$$Y \vdash_B b\ \&\ \forall b' \in Y \exists a' \in A[a \vdash_A a'\ \&\ (a', b',) \in R],$$

that is,

$$Y \vdash_B b\ \&\ \forall b' \in Y \exists a' \in A[a \vdash_A a'\ \&\ b' \in f(\overline{a'})].$$

Clearly, $Y \subseteq f(\overline{a})$, and we have $b \in f(\overline{a})$. Moreover, the quasi-linear function $\mathcal{F}R$ determined by R is actually equal to f. To see this, let x be an element of $\underline{A}$. We have

$$\begin{aligned} b \in \mathcal{F}(R)(x) &\Leftrightarrow \exists a \in x\ (a, b) \in R \\ &\Leftrightarrow \exists a \in x\ b \in f(\overline{a}) \\ &\Leftrightarrow b \in f(x)\ (\text{ since f is quasi-linear}). \end{aligned}$$

Suppose $R,\ S :\ \underline{A} \rightarrow \underline{B}$ are linear approximable mappings such that $\mathcal{F}R$ equals $\mathcal{F}S$. To show that $\mathcal{F}$ is faithful, it suffices to prove $R = S$. For that purpose, let (a, b) be a member of R. We have

$$\begin{aligned} (a, b) \in R &\Rightarrow b \in \mathcal{F}(R)(\overline{a}) \\ &\Rightarrow b \in \mathcal{F}(S)(\overline{a}) \\ &\Rightarrow \exists a' \in \overline{a}\ (a', b) \in S \\ &\Rightarrow \exists a'\ a \vdash_A a'\ \&\ (a', b) \in S \\ &\Rightarrow (a, b) \in S. \end{aligned}$$

By symmetry, we also have $S \subseteq R$. Therefore, $S = R$.

□

For irreducible information systems

$$\underline{A} = (A, Con_A, \vdash_A)\ \text{ and }\ \underline{B} = (B, Con_B, \vdash_B),$$

we introduce the following constructions:

$$\begin{aligned}
&\underline{A} \otimes \underline{B} = (A \times B, Con_{A\otimes B}, \vdash_{A\otimes B}), \quad \text{where} \\
&\qquad X \in Con_{A\otimes B} \ \text{ iff } \ \pi_A(X) \in Con_A \ \& \ \pi_B(X) \in Con_B, \\
&\qquad X \vdash_{A\otimes B} (a,b) \ \text{iff } \ \pi_A(X) \vdash_A a \ \& \ \pi_B(X) \vdash_B b; \\
&\underline{A} \multimap \underline{B} = (A \times B, Con_{A \multimap B}, \vdash_{A \multimap B}), \quad \text{where} \\
&\qquad X \in Con_{A \multimap B} \ \text{ iff } \ \pi_A(X) \in Con_A \Rightarrow \pi_B(X) \in Con_B, \\
&\qquad X \vdash_{A \multimap B} (a,b) \ \text{iff } \ \{b' \mid \exists a'.a \vdash_A a' \ \& \ (a',b') \in X\} \vdash_B b; \\
&\underline{A} \times \underline{B} = (\{0\} \times A \cup \{1\} \times B, Con_{A\times B}, \vdash_{A\times B}), \quad \text{where} \\
&\qquad X \in Con_{A\times B} \ \text{ iff } \ \{a \mid (0,a) \in X\} \in Con_A \ \& \ \{b \mid (1,b) \in X\} \in Con_B, \\
&\qquad X \vdash_{A\times B} (i,p) \ \text{iff } (i = 0 \ \& \ \{a \mid (0,a) \in X\} \vdash_A p) \\
&\qquad\qquad\qquad \text{or } (i = 1 \ \& \ \{b \mid (1,b) \in X\} \vdash_B p) \\
&\bot = (\{\bullet\}, \{\emptyset, \{\bullet\}\}, \{(\bullet,\bullet)\}).
\end{aligned}$$

All these constructions preserve irreducible information systems, as confirmed by the following theorem. It is useful to note that for the quasi-linear function space, we have

$$\begin{aligned}
X \vdash_{A \multimap B} (a,b) &\Leftrightarrow \{b' \mid \exists a'.a \vdash_A a' \ \& \ (a',b') \in X\} \vdash_B b \\
&\Leftrightarrow \exists X' \subseteq X \ a \vdash_A \pi_A(X') \ \& \ \pi_B(X') \vdash_B b.
\end{aligned}$$

Theorem 41 *If $\underline{A}$ and $\underline{B}$ are irreducible information systems, then $\underline{A} \otimes \underline{B}$, $\underline{A} \multimap \underline{B}$, $\underline{A} \times \underline{B}$ are irreducible information systems.*

PROOF. It is easy to see that $\otimes$, $\multimap$, and $\times$ preserves information systems. We show that they preserve irreducible information systems.

$\otimes$: If $\underline{A}$ and $\underline{B}$ do not contain redundant tokens, then neither does $\underline{A} \otimes \underline{B}$, because by the definition of the entailment relation $\vdash_{A\otimes B}$, any redundant token in the tensor product will be reduced to redundant tokens in the components.

$\multimap$: Note that for a non-redundant token b (or in fact any token considered in this paper), we do not have $\emptyset \vdash b$. So, if

$$X \vdash_{A \multimap B} (a,b) \ \text{ and } \ (a,b) \vdash_{A \multimap B} (a',b')$$

for each (a',b') in X, then $G \vdash_B b$ and $b \vdash g$ for each g in G, where G stands

for the non-empty set

$$\{b' \mid \exists a'.a \vdash_A a' \ \& \ (a', b') \in X\}.$$

This means any redundant token (a, b) in the quasi-linear function space will be reduced to a redundant token b in $\underline{B}$.

$\times$: This case is straightforward.

□

There is a one-one correspondence between the set of quasi-linear approximable mappings $R : \underline{A} \to \underline{B}$ and the elements $\mid \underline{A} \multimap \underline{B} \mid$.

Lemma 42 *Let $\underline{A}$ and $\underline{B}$ be irreducible information systems. A relation*

$$R \subseteq A \times B$$

is a quasi-linear approximable mapping $R : \underline{A} \to \underline{B}$ if and only if it is an ideal element of $\underline{A} \multimap \underline{B}$.

PROOF. The conditions related to consistency are the same for a quasi-linear approximable mapping and for an ideal element in the quasi-linear function space. Let $R \subseteq A \times B$. We have, as far as the other condition is concerned,

$$\begin{aligned}
R \in \mid \underline{A} \multimap \underline{B} \mid \Leftrightarrow & \text{ for any finite subset } X \text{ of } R, \\
& \quad \text{if } X \vdash (a, b), \text{ then } (a, b) \in R \\
\Leftrightarrow & \text{ for any finite subset } X \text{ of } R, \\
& \quad \text{if } \{b' \mid \exists a' \ (a \vdash_A a' \ \& \ (a', b') \in X)\} \vdash_B b, \\
& \quad \text{then } (a, b) \in R \\
\Leftrightarrow & \text{ for any finite subset } X' \text{ of } R, \\
& \quad \text{if } a \vdash_A \pi_A(X') \ \& \ \pi_B(X') \vdash_B b, \\
& \quad \text{then } (a, b) \in R \\
\Leftrightarrow & \text{ for any finite subset } Y \text{ of } B \text{ (let } Y = \pi_B X'), \\
& \quad \text{if } Y \vdash_B b \ \& \ \forall b' \in Y \exists a' \in A \ [a \vdash a' \ \& \ (a', b') \in R] \\
& \quad \text{then } (a, b) \in R \\
\Leftrightarrow & \ R \text{ is a quasi-linear approximable mapping.}
\end{aligned}$$

□

Theorem 43 **IRI.ql** *and* **IRL.ql** *are symmetric monoidal closed categories with finite product.* **IRI.a** *and* **IRL.a** *are cartesian closed categories.*

PROOF. We present the proof that **IRI.ql** is monoidal closed and **IRI.a** is cartesian closed, with some of the details omitted.

First, the function $\tau : (A \times B) \times C \to A \times (B \times C)$ specified by

$$((a, b), c) \longmapsto (a, (b, c))$$

witnesses an isomorphism

$$(\underline{A} \otimes \underline{B}) \multimap \underline{C} \cong \underline{A} \multimap [\underline{B} \multimap \underline{C}].$$

In light of Lemma 42, this shows there is a natural isomorphism between $\mathrm{Hom}(\underline{A} \otimes \underline{B}, \underline{C})$ and $\mathrm{Hom}(\underline{A}, \underline{B} \multimap \underline{C})$. Thus $(-) \otimes B$ is a left adjoint of $B \multimap (-)$.

It is routine to show that

$$u \in |\, \underline{A} \times \underline{B} \,| \text{ if and only if } \{a \mid (0, a) \in u\} \in |\, \underline{A} \,| \quad \text{and } \{b \mid (1, b) \in u\} \in |\, \underline{B} \,| .$$

Thus

$$|\, \underline{A} \times \underline{B} \,| \cong |\, \underline{A} \,| \times |\, \underline{B} \,| .$$

Since the cartesian product is the product in **QPA.c**, and **IRI.a** and **QPA.c** are equivalent, $\times$ is the product in **IRI.a**.

We now show that

$$\begin{aligned} \underline{A} \to \underline{B} = (&Con_A \times B, Con_{A \to B}, \vdash_{A \to B}), \quad \text{where} \\ &X \in Con_{A \to B} \text{ iff } \bigcup \pi_A(X) \in Con_A \Rightarrow \pi_B(X) \in Con_B, \\ &X \vdash_{A \to B} (u, b) \text{ iff } \{b' \mid \exists u'. u \vdash_A u' \ \&\ (u', b') \in X\} \vdash_B b \end{aligned}$$

is the exponential. According to this construction, if

$$X \vdash_{A \to B} (u, b) \text{ and } (u, b) \vdash_{A \to B} X,$$

then b would be redundant, which is impossible. Thus $\underline{A} \to \underline{B}$ is an irreducible information system. It is easy to check that

$$\underline{A} \times \underline{B} \to \underline{C} \cong \underline{A} \to [\underline{B} \to \underline{C}],$$

since we have the structure-preserving one-one correspondence

$$(\{0\} \times X \cup \{1\} \times Y, c) \longmapsto (X, (Y, c)).$$

Therefore $(-) \times B$ is a left adjoint of $B \to (-)$.

□

As a consequence of the previous theorem, **QPA.ql** and **QPL.ql** are monoidal closed categories, and **QPA.c** and **QPL.c** are cartesian closed categories.

The decomposition $\underline{A} \to \underline{B} \cong !\underline{A} \multimap \underline{B}$ holds for both irreducible information systems and conflict-free irreducible information systems. For a irreducible information system $\underline{A}$, we have

$$\begin{aligned} !\underline{A} = (\mathit{Con}_A, \mathit{Con}_{!A}, \vdash_{!A}), \quad & \text{where} \\ X \in \mathit{Con}_{!A} \ \text{ iff } \ & \bigcup X \in \mathit{Con}_A, \\ X \vdash_{!A} v \ \text{iff } \ & \exists u \in X.\, u \vdash_A v. \end{aligned}$$

Theorem 44 *Let $\underline{A}$, $\underline{B}$ be irreducible information systems. Then*

$$\underline{A} \to \underline{B} \cong !\underline{A} \multimap \underline{B}.$$

PROOF. For Con, we have

$$\begin{aligned} X \in \mathit{Con}_{A \to B} &\Leftrightarrow \text{if } \bigcup \pi_A(X) \in \mathit{Con}_A \text{ then } \pi_B(X) \in \mathit{Con}_B \\ &\Leftrightarrow X \subseteq \mathit{Con}_A \times B \\ &\quad \& \text{ if } \pi_{!A}(X) \in \mathit{Con}_{!A} \text{ then } \pi_B(X) \in \mathit{Con}_B \\ &\Leftrightarrow X \in \mathit{Con}_{!A \multimap B}. \end{aligned}$$

For $\vdash$, we have

$$\begin{aligned} X \vdash_{A \to B} (u, b) &\Leftrightarrow \{b' \mid \exists u' \, (u \vdash_A u' \,\&\, (u', b') \in X)\} \vdash_B b \\ &\Leftrightarrow \exists X' \subseteq X \; u \vdash_A \bigcup \pi_A(X') \,\&\, \pi_B X' \vdash_B b \\ &\Leftrightarrow \exists X' \subseteq X \; u \vdash_A \pi_{!A}(X') \,\&\, \pi_B X' \vdash_B b \\ &\Leftrightarrow \{b' \mid \exists u' \, (u \vdash_{!A} u' \,\&\, (u', b') \in X)\} \vdash_B b \\ &\Leftrightarrow X \vdash_{!A \multimap B} (u, b). \end{aligned}$$

□

Note that **IRI.ql** does not have a dualizing object. This is because $\underline{A} \multimap \bot$ is always conflict-free, but $\underline{A}$ may not be. However, **IRL.ql** does not have a

dualizing object either because of the following observation.

Proposition 45 *Let $\underline{A}$ be a conflict-free, irreducible information system. Then*

$$(|\ \underline{A} \multimap \perp\ |, \subseteq)$$

is a prime algebraic lattice.

PROOF. According to the definitions about conflict-free, irreducible information systems, we have

$$\underline{A} \multimap \perp = (A \times \{\bullet\}, \vdash_A \multimap_\perp), \quad \text{where}$$
$$X \vdash_A \multimap_\perp (a, b) \text{ iff } \{b' \mid \exists a'.a \vdash_A a' \ \&\ (a', b') \in X\} \vdash_\perp b.$$

Since the second component of a token can only be the $\bullet$, we have

$$X \vdash_A \multimap_\perp (a, \bullet) \text{ iff } \{\bullet \mid \exists a'.a \vdash_A a' \ \&\ (a', \bullet) \in X\} \vdash_\perp \bullet,$$

or

$$X \vdash_A \multimap_\perp (a, \bullet) \text{ iff } \exists (a', \bullet) \in X\ (a', \bullet) \vdash_A \multimap_\perp (a, \bullet).$$

The property

$$Y \vdash b \Leftrightarrow \exists b' \in Y\ b' \vdash b$$

implies for each token $a \in A$, $\overline{a}$, as an ideal element, is a complete prime in $(|\ \underline{A}\ |, \subseteq)$. Therefore, $(|\ \underline{A}\ |, \subseteq)$ is a prime algebraic lattice.

□

The good news is that **IRL.a** has coproducts, which are different from products. The construction of gluing the bottoms and adding a new top to the disjoint copies of two lattices preserves quasi-prime algebraic lattices.

Part III. A Saturated Domain

We have shown that quasi-prime algebraic domains with quasi-linear functions form a monoidal closed category. The unique characteristic of the category is that the morphisms are not "linear", as the term "quasi-linear" suggests. This is a bit unexpected, since all other known domain theoretic linear categories use linear functions as morphisms.

However, how robust and useful the concept of quasi-prime algebraic domains is depends on whether or not they have other nice domain theoretic properties. One of the desirable properties is the existence of a universal (or even saturated) domain in a certain category. The other related property to have is a framework for solving domain equations by fixed point construction, as in [11,18]. It is the purpose of this part to establish these results for irreducible information systems, which represent quasi-prime algebraic domains.

One of the most useful results on universal domains is given in the work of Droste and Göbel [3], who introduced the Fraissé-Jónsson theorem in model theory into the area of domain theory. This makes it much easier to show the existence of certain universal domains because it reduces the existence of a saturated structure to the amalgamation property of the finite objects of a certain category.

We apply the result of Droste and Göbel for showing the existence of a saturated (universal, homogeneous) quasi-prime algebraic domain. Our main definition here is the notion of q-embeddings for quasi-prime algebraic domains. These are the usual embedding-projection pairs with the additional property that quasi-primes are preserved by embedding. The appropriate notion of embeddings for Scott domains (call them s-embeddings) [11,4] and for dI-domains (call them r-embeddings – "r" for rigid) [18] are well-known . However, none of the these embeddings works for quasi-prime algebraic domains, for the following reasons:

- The s-embeddings are too general: under this embedding the colimit of an ω-chain of finite Scott domains (which are quasi-prime algebraic) need not be quasi-prime algebraic, because any Scott domain can be seen as a colimit of this kind.
- The r-embeddings are too specific: there are certain quasi-prime algebraic domains which cannot be represented as a colimit of any chain of finite Scott domains, although the r-embeddings are suitable for the I-domains (dI-domains without axiom d).

The important property about q-embeddings is that quasi-primes are preserved. However, irreducible information systems make it possible to capture the notion of q-embeddings naturally as standard embeddings between infor-

mation systems. We work in the framework of irreducible information systems and embeddings, a desired algebroidal category. The finite objects of the category are shown to have the amalgamation property. This implies the existence of a saturated irreducible information system.

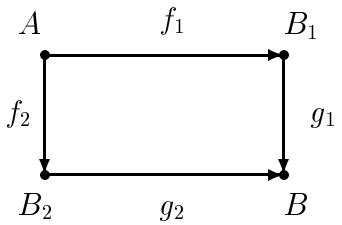
Based on embeddings on irreducible information systems, we will also introduce a cpo of irreducible information systems on which various constructions are shown to induce continuous functions. This implies the existence of recursively defined irreducible information systems. Finally, we briefly illustrate how this technique can be applied to construct a model for the un-typed lambda calculus.

9 Universality and Amalgamation

A unified theory of universal objects can be found in [3]. We review the key notions of that theory in this section. The basic theorem is that in any algebroidal category in which all morphisms are monic, the existence of a universal, homogeneous object is equivalent to the amalgamation property of the (finite objects of the) category. Let us recall some of the relevant definitions now.

Let $\mathbf{C}$ be a category where all the morphisms are monic (corresponding to the intuitive notion of one-to-one). Let $\mathbf{C}_f$ be the finite objects of $\mathbf{C}$. An object U of $\mathbf{C}$ is universal in $\mathbf{C}$ (or, $\mathbf{C}$-universal) if for any object A in $\mathbf{C}$, there is an arrow $f : A \to U$. U is homogeneous (or, $\mathbf{C}_f$-homogeneous) if for any finite object A with a pair of arrows $f, g : A \to U$, there is an isomorphism $h : U \to U$ such that $f = h \circ g$. U is saturated if for any A, B of $\mathbf{C}_f$ and arrows $f : A \to U$, $g : A \to B$, there is an $h : B \to U$ such that $h \circ g = f$.

A category $\mathbf{C}$ is said to have the amalgamation property if for any arrows $f_1 : A \to B_1$, $f_2 : A \to B_2$ in $\mathbf{C}$, there are arrows $g_1 : B_1 \to B$, $g_2 : B_2 \to B$ in $\mathbf{C}$ such that the following diagram commute.



The result of Droste and Göbel is based on the notion of an algebroidal cate-

gory.

Definition 46 *An algebroidal category is one which has the following properties:*

(i) It has an initial object,
(ii) Every object of the category is a colimit of an ω-chain of finite objects,
(iii) Every ω-chain of finite objects has a colimit, and
(iv) The number of (up to isomorphism) finite objects is countable.

Theorem 47 *(Droste and Göbel) Let $\mathbf{C}$ be an algebroidal category with all morphisms monic. Let $\mathbf{C}_f$ be the full subcategory of finite objects of $\mathbf{C}$. The existence of a $\mathcal{C}$-universal, $\mathbf{C}_f$-homogeneous object is equivalent to the amalgamation property of $\mathbf{C}_f$, which is in turn equivalent to the existence of a $\mathbf{C}_f$ saturated object.*

Note that in various categories of information systems, finite objects are exactly those with a finite token set.

10 Embeddings on Irreducible Information Systems

To be able to apply Theorem 47, we need to introduce an algebroidal category of irreducible information systems. The morphisms for this category are the standard embeddings on information systems.

Definition 48 *Let $\underline{A} = (A, Con_A, \vdash_A)$, $\underline{B} = (B, Con_B, \vdash_B)$ be irreducible information systems. A function $f : A \to B$ is an embedding of $\underline{A}$ into $\underline{B}$ if*

1. *f is one-to-one*;
2. $\forall X \subseteq A \forall a \in A$
$$X \in Con_A \Leftrightarrow f(X) \in Con_B,$$
$$X \vdash_A a \Leftrightarrow f(X) \vdash_B f(a).$$

It is clear that irreducible information systems with embeddings form a category, which we written as $\mathbf{Q}$. We present several propositions leading to the main conclusion that $\mathbf{Q}$ is algebroidal.

Proposition 49 *Colimits exist in $\mathbf{Q}$ for ω-chains of finite irreducible information systems.*

PROOF. Let $f_i : \underline{A}_i \to \underline{A}_{i+1}, i \in \omega$, be an ω-chain of finite irreducible information systems. Without loss of generality, suppose $A_i \subseteq A_{i+1}$ for $i \in \omega$. Let $\underline{A} = (A, Con, \vdash)$, where $A = \bigcup_{i\in\omega} A_i$, $\vdash = \bigcup_{i\in\omega} \vdash_i$, and $Con = \bigcup_{i\in\omega} Con_i$. $\underline{A}$ is again an irreducible information system, for any redundant token in A must already be redundant in some $\underline{A}_i$. It is clear that the inclusion from A_i to A is an embedding. This makes it clear that $\underline{A}$ is the colimit of the chain $\underline{A}_i$, since $\underline{A}$ is irreducible.

□

Proposition 50 *Every irreducible information system is the colimit of an ω-chain of finite information systems in* **Q**.

PROOF. With out loss of generality, let $\underline{A} = (A, Con, \vdash)$ be an irreducible information system whose token set is written as $A := \{a_i \mid i \in \omega\}$. Inductively, let $\underline{A}_0$ be the empty information system, and $\underline{A}_i = (A_i, Con_i, \vdash_i)$ be the finite information system such that A_i consists of $A_{i-1} \cup \{a_{i-1}\}$. Let $Y \vdash_i b$ iff $Y \vdash b$ and $Y \in Con_i$ iff $Y \in Con$.

Each $\underline{A}_i$ is a finite, irreducible information system. Moreover, the inclusion from $\underline{A}_i$ to $\underline{A}_{i+1}$ is clearly an embedding for each $i \in \omega$. It is easy to see that $\underline{A}$ is the colimit of the chain $\{\ \underline{A}_i \mid i \in \omega\}$.

□

The above propositions, together with the observation that the empty information system is initial, implies the following.

Theorem 51 *Irreducible information systems with embeddings form an algebroidal category.*

It is easy to see that corresponding to embeddings there is a notion of embedding-projection pairs (*i.e.*, q-embedding-projection pairs) on quasi-prime algebraic domains. These are just the usual embedding-projection pairs with the additional requirement that they preserve quasi-primes.

11 A Saturated Quasi-Prime Algebraic Domain

The purpose of this section is to show that the finite objects of the category **Q** have the amalgamation property. In light of Theorem 47 and Theorem 51, this means there exists a saturated irreducible information system. Note that our proof below follows the style of [4].

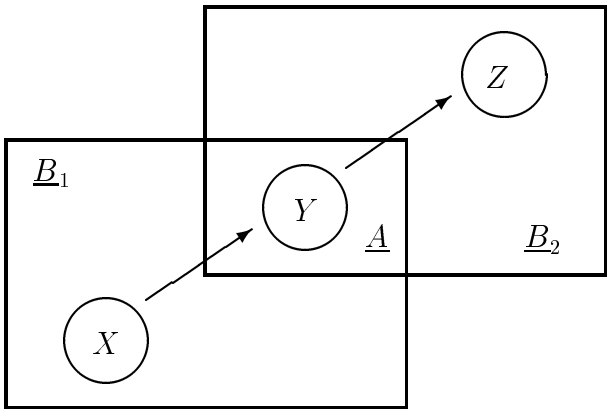


Fig. 5. $\underline{A}$ acting as a bridge between $\underline{B}_1$ and $\underline{B}_2$

Let $\underline{A} = (A, Con_A, \vdash_A)$, $\underline{B}_1 = (B_1, Con_1, \vdash_1)$ and $\underline{B}_2 = (B_2, Con_2, \vdash_2)$ be finite irreducible information systems such that $f_1 : \underline{A} \to \underline{B}_1$ and $f_2 : \underline{A} \to \underline{B}_2$ are embeddings. By renaming the tokens, it is enough to consider the case where f_i's are inclusions (identities) and $A = B_1 \cap B_2$.

Definition 52 *We construct the information system* $\underline{B} = (B, Con, \vdash)$ *from* $\underline{B}_1$ *and* $\underline{B}_2$*, where*

- $B := B_1 \cup B_2$;
- $X \in Con$ *if and only if*

$$\exists Z \exists Y \supseteq X \cap B_1.\ [Y \vdash_1 Z \vdash_2 X \cap B_2] \quad \textit{or}$$
$$\exists Z \exists Y \supseteq X \cap B_2.\ [Y \vdash_2 Z \vdash_1 X \cap B_1];$$

- $\vdash := \bigcup_{i \geq 0} \vdash^i$*, where* $\vdash^i$*'s are specified as follows:*

$$\vdash^0 = \{(X, a) \mid X \in Con \ \&\ \textit{either } B_1 \cap X \vdash_1 a \ \textit{ or } \ B_2 \cap X \vdash_2 a\},$$
$$\vdash^{i+1} = \{(X, a) \mid \exists Y.\ X \vdash^i Y \vdash^i a\}.$$

Before getting into technical details, we would like to give the reader an intuitive feeling of what is going on. $\underline{B}_1$ and $\underline{B}_2$ are two given structures sharing a common substructure $\underline{A}$. The question is whether $\underline{B}_1$ and $\underline{B}_2$ can both be seen living in a larger structure. The larger structure must respect the already-existing relationships among $\underline{B}_1$, $\underline{B}_2$, and $\underline{A}$. In particular, $\underline{A}$ serves as a "bridge" between $\underline{B}_1$ and $\underline{B}_2$, so some X from B_1 may entail some Y from A, which may in turn entail some Z in B_2. Entailment must be transitive; so in the larger structure we should have X entail Z (see Figure 5). That is why our construction for $\vdash$ is a kind of transitive closure.

The consistency predicate *Con*, however, must come before $\vdash$ can be defined:

the $\vdash$ relation is only defined on consistent sets. On the surface, it may look that there can be several choices for specifying Con. Letting $X \in Con$ if and only if $X \cap B_1 \in Con_1$ and $X \cap B_2 \in Con_2$, for example, may seem to be a reasonable choice. A second thought, however, reveals that this is not the case.

Example 53 *Let $A = \{1, 2\}$, with a trivial consistency predicate and a trivial entailment relation. Let $B_1 = \{\alpha, 1, 2\}$, with $\alpha \vdash_1 1$, and let $B_2 = \{1, 2, a, b\}$ such that $\{a, b\} \notin Con_2$, and $\{1, 2\} \vdash_2 b$, $a \vdash_2 2$. It is easy to check that we have three irreducible information systems in this way, and A embeds into B_1 and B_2. In the bigger system to be constructed, do we want to have $\{\alpha, a\} \in Con$ because $\{\alpha\} \in Con_1$ and $\{a\} \in Con_2$? The answer is "no": the given conditions $\alpha \vdash_1 1$, $a \vdash_2 2$, and $\{1, 2\} \vdash_2 b$ would imply $\{\alpha, a\} \vdash \{a, b\}$, which must be inconsistent.*

Our original specification in Definition 52 seems to be the only right choice. We now check that $\underline{B}$ is indeed a irreducible information system which makes the diagram required by the amalgamation commute. This is achieved by a sequence of lemmas. The reader is advised to be careful about subscripts and supscripts, though, by keeping in mind that subscripts represent given relations while supscripts represent relations used in the process for constructing $\underline{B}$.

Lemma 54 *The structure $\underline{B}$ given in Definition 52 is an information system.*

PROOF. We show that $\underline{B}$ satisfies the axioms of an information system.

1. $(X \subseteq Y \in Con) \Rightarrow X \in Con$: This is straightforward.

2. $b \in B \Rightarrow \{b\} \in Con$: If $b \in B_1 - B_2$, then

$$\{b\} \vdash_1 \emptyset \vdash_2 \{b\} \cap B_2,$$

which means, by definition, $\{b\} \in Con$. If $b \in B_1 \cap B_2$, then

$$\{b\} \vdash_1 \{b\} \vdash_2 \{b\} \cap B_2,$$

which again shows $\{b\} \in Con$.

3. $X \vdash a \Rightarrow X \cup \{a\} \in Con$: By mathematical induction on i, we show that each $\vdash^i$ has this property. When $i = 0$, $X \vdash^0 a$ implies two possible yet similar cases, the first being $B_1 \cap X \vdash_1 a$. Since $X \in Con$, there exist Y and Z such that $Y \vdash_1 Z \vdash_2 X \cap B_2$, with $Y \supseteq X \cap B_1$, or $Y \vdash_2 Z \vdash_1 X \cap B_1$, with $Y \supseteq X \cap B_2$. We have $Y \vdash_1 a$ and $Y \cup \{a\} \in Con_1 \subseteq Con$. Hence

$$Y \cup \{a\} \supseteq (X \cup \{a\}) \cap B_1.$$

If $a \notin B_2$, then

$$Y \cup \{a\} \vdash_1 Z \vdash_2 (X \cup \{a\}) \cap B_2;$$

if $a \in B_2$, then

$$Y \cup \{a\} \vdash_1 Z \cup \{a\} \vdash_2 (X \cup \{a\}) \cap B_2.$$

This shows $X \cup \{a\} \in \mathit{Con}$. For the induction step, let $X \vdash^{i+1} a$. By definition, there is some Y such that $X \vdash^i Y \vdash^i a$. Note that $\vdash^k$ has the property that $X \subseteq Y \in \mathit{Con}$ and $X \vdash^k a$ implies $Y \vdash^k a$ for each $k \geq 0$, We conclude that $X \cup Y \cup \{a\} \in \mathit{Con}$ since $X \cup Y \in \mathit{Con}$ and $X \cup Y \vdash^i a$.

4. $(X \in \mathit{Con}\ \&\ a \in X) \Rightarrow X \vdash a$: This is because either $a \in B_1$ in which case $B_1 \cap X \vdash_1 a$, or $a \in B_2$ in which case $B_2 \cap X \vdash_2 a$. we also use the fact that $\vdash^i \subseteq \vdash^{i+1}$ for each $i \geq 0$ (this can be seen by taking $Y = X$ in Definition 52).

5. $(X \vdash Y\ \&\ Y \vdash c) \Rightarrow X \vdash c$: Use the monotonicity of $\vdash^i$ in i.

□

Our next lemma is the key to the proofs of various results later.

Lemma 55 *For the information system $\underline{B}$ given in Definition 52, $X \subseteq B_1$ and $X \vdash Y$ imply*

$$X \vdash_1 Y \cap B_1 \quad \textit{and}$$
$$\exists Z \subseteq A.\ X \vdash_1 Z \vdash_2 Y \cap B_2.$$

PROOF. We prove this lemma by mathematical induction on k, with the understanding that $\vdash := \bigcup_{k \geq 0} \vdash^k$, as given in Definition 52.

The case $k = 0$ has to be checked separately. When $X \vdash^0 Y$ we have, by definition, either $X \cap B_1 \vdash_1 b$, or $X \cap B_2 \vdash_2 b$ for each $b \in Y$ (Note that b can be in $B_2 - B_1$, even though $X \subseteq B_1$). For each b in $Y \cap B_1$, it must be the case that $X \vdash_1 b$; So $X \vdash_1 Y \cap B_1$. Let

$$Z = [X \cap B_2] \cup [Y \cap B_1 \cap B_2].$$

Clearly we have $X \vdash_1 Z$. Note that

$$Y \cap B_2 = [Y \cap (B_2 - B_1)] \cup [Y \cap B_2 \cap B_1].$$

If $b \in Y \cap (B_2 - B_1)$, then we must have $X \cap B_2 \vdash_2 b$. If $b \in Y \cap B_1 \cap B_2$, then $Y \cap B_1 \cap B_2 \vdash_2 b$. Therefore,

$$X \vdash_1 Z \vdash_2 Y \cap B_2.$$

Base case: $k = 1$. When $X \vdash^1 Y$, it can be seen from definition that there is some G such that

$$X \vdash^0 G \vdash^0 Y.$$

By the previous case, $X \vdash^0 G$ implies the existence of some $Q \subseteq A$ such that

$$X \vdash_1 G \cap B_1 \;\&\; X \vdash_1 Q \vdash_2 G \cap B_2.$$

Let $R := (G \cap B_1) \cup Q$. We have $X \vdash_1 R$ and $R \subseteq B_1$. It is crucial to note that $R \vdash^0 Y$. This is because for each $b \in Y$, either $G \cap B_1 \vdash_1 b$, or $G \cap B_2 \vdash_2 b$. For the first case we have $R \vdash_1 b$, and for the latter, we have $R \vdash_2 Q \vdash_2 G \cap B_2 \vdash_2 b$ and, therefore, $R \vdash_2 b$. By the result for the case $k = 0$ again, $R \vdash^0 Y$ implies the existence of some $S \subseteq A$ such that

$$R \vdash_1 Y \cap B_1 \;\&\; R \vdash_1 S \vdash_2 Y \cap B_2.$$

Therefore, we have

$$X \vdash_1 Y \cap B_1 \;\&\; X \vdash_1 S \vdash_2 Y \cap B_2,$$

as required for the case $i = 1$.

Induction step. Suppose the result holds for k, where $k \geq 1$. If $X \vdash^{k+1} Y$, then $X \vdash^k G \vdash^k Y$ for some $G \subseteq B_1 \cup B_2$. By the induction hypothesis, there is some $Q \subseteq A$ such that

$$X \vdash_1 G \cap B_1 \;\&\; X \vdash_1 Q \vdash_2 G \cap B_2.$$

Let $R := (G \cap B_1) \cup Q$, a consistent subset of B_1. We have $R \vdash^0 G \vdash^k Y$. This means $R \vdash^0 G \vdash^{k-1} Y' \vdash^{k-1} Y$, or $R \vdash^k Y' \vdash^{k-1} Y$, for some Y'. By the induction hypothesis again, we have

$$R \vdash_1 Y' \cap B_1 \;\&\; R \vdash_1 S \vdash_2 Y' \cap B_2$$

for some $S \subseteq A$. Now let $T := (Y' \cap B_1) \cup S$, a consistent subset of B_1. We have $T \vdash^0 Y' \vdash^{k-1} Y$, or $T \vdash^k Y$. We use induction hypothesis once more to get

$$T \vdash_1 Y \cap B_1 \;\&\; T \vdash_1 U \vdash_2 Y \cap B_2$$

for some $U \subseteq A$.

Therefore, we have

$$X \vdash_1 Y \cap B_1 \;\&\; X \vdash_1 U \vdash_2 Y \cap B_2.$$

□

Of course, a symmetric version of Lemma 55 also holds, with X a subset of B_2. This lemma should be helpful to a direct definition for the entailment $\vdash$

in Definition 52. Given that $\underline{B}$ is a finite system, one wonders why $\vdash$ has to be defined in a inductive manner. However, we don't have a better definition available at present.

There are a couple of ways Lemma 55 can be interpreted. It says that for X's from B_1, $\vdash$ is the same as $\vdash^1$. This means when X is restricted to B_1, $X \vdash Y$ if and only if $X \vdash^1 Y$. It also says that

$$X \subseteq B_1 \ \& \ X \vdash Y \Rightarrow \exists X'.\ X \vdash_1 X' \vdash^0 Y,$$

by taking X' to be $(Y \cap B_1) \cup Z$. These corollaries are sometimes more handy to use.

Lemma 56 *For $X \subseteq B_1$ and $a \in B_1$,*

$$X \in \mathit{Con}_1 \Leftrightarrow X \in \mathit{Con} \ \ \mathit{and} \ \ X \vdash_1 a \Leftrightarrow X \vdash a.$$

PROOF. If $X \in \mathit{Con}_1$ then by taking $Z = X \cap B_2$ we have

$$X \vdash_1 Z \vdash_2 X \cap B_2.$$

Therefore, by Definition 52, $X \in \mathit{Con}$.

Suppose $X \in \mathit{Con}$ and $X \subseteq B_1$. By Definition 52, either

$$Y \vdash_1 Z \vdash_2 X \cap B_2$$

for some $Y \supseteq X \cap B_1$, or

$$Y \vdash_2 Z \vdash_1 X \cap B_1$$

or some $Y \supseteq X \cap B_2$. The first case implies $Y \in \mathit{Con}_1$ and hence, so does X. For the second case, we have $Z \vdash_1 X$ and so $X \in \mathit{Con}_1$.

It is clear that $X \vdash_1 a$ implies $X \vdash a$. The reverse follows immediately from Lemma 55.

□

We now come to another important lemma which shows that $\underline{B}$ is irreducible. (It is clearly finite.)

Lemma 57 *The information system $\underline{B}$ specified in Definition 52 is irreducible.*

PROOF. We show that for any a in $B_1 \cup B_2$, $X \dashv\vdash a$ in $\underline{B}$ happens only when a is already a member of X, where $X \dashv\vdash a$ is an abbreviation for $X \vdash a$ and $a \vdash b$ for every b in X.

Without loss of generality, suppose a is a member of B_1. It suffices to show that

$$X \dashv\vdash^i a \Rightarrow a \in X$$

for $i \geq 0$. This is done by mathematical induction on i.

Base case: $i = 0$. We have $X \vdash^0 a$ and $a \vdash^0 X$. By Lemma 55, there is some $Z \subseteq A$ such that

$$a \vdash_1 X \cap B_1 \;\&\; a \vdash_1 Z \vdash_2 X \cap B_2.$$

Note that $X \vdash^0 a$ means either $X \cap B_1 \vdash_1 a$, or $X \cap B_2 \vdash_2 a$. The first case implies $X \cap B_1 \dashv\vdash_1 a$, so $a \in X \cap B_1$ since a is a non-redundant token of $\underline{B}_1$. For the second case, we have $Z \vdash_2 a$, which implies $Z \vdash_1 a$ because both a and Z are form B_1. In this case we have $Z \dashv\vdash_1 a$ and hence $a \in Z$. Note that a is a token of $\underline{A}$, which implies it is also a non-redundant token of $\underline{B}_2$. However, we now have $X \cap B_2 \dashv\vdash_2 a$, so $a \in X \cap B_2$.

Before the induction step, we prove this useful fact:

$$\text{If } a \in B_1 \text{ and } X \dashv\vdash a, \text{ then either } \; a \vdash^0 X, \;\; \text{or } \; a \in X.$$

Indeed, because $a \vdash X$ and $a \in B_1$, we have, by Lemma 55,

$$a \vdash_1 X \cap B_1 \;\&\; a \vdash_1 Z_1 \vdash_2 X \cap B_2$$

for some $Z_1 \subseteq A$. Let $Y_1 = (X \cap B_1) \cup Z_1$. By repeated application of Lemma 55, we have

$$\begin{aligned} a \vdash_1 Y_1 \vdash^0 X \vdash a &\Rightarrow a \vdash_1 Y_1 \vdash a \\ &\Rightarrow \exists Y_2 \subseteq B_1.\, a \vdash_1 Y_1 \vdash_1 Y_2 \vdash^0 a \;\; (\text{ note that } a \in B_1) \\ &\Rightarrow a \vdash_1 Y_1 \vdash_1 Y_2 \vdash_1 a \\ &\Rightarrow a \in Y_1. \end{aligned}$$

This means either $a \in X \cap B_1$, or $a \in Z_1$. If $a \in X \cap B_1$, we are done; so assume $a \in Z_1$. We have

$$a \vdash_1 X \cap B_1 \;\text{ and }\; a \vdash_2 Z_1 \vdash_2 X \cap B_2.$$

Therefore, $a \vdash^0 X$.

Now let's return to the induction step. Suppose $X \dashv\vdash^k a$ for some $a \in B_1$ and $k \geq 1$. By the previous observation, we have either $a \vdash^0 X$, or $a \in X$. If $a \in X$, there is nothing more to prove; so we assume $a \vdash^0 X$.

Note that $X \vdash^k a$ implies $X \vdash^{k-1} G \vdash^{k-1} a$ for some G. Since we have $a \vdash G$,

Lemma 55 implies

$$a \vdash_1 G \cap B_1 \;\&\; a \vdash_1 Z_2 \vdash_2 G \cap B_2$$

for some $Z_2 \subseteq A$. Let $Y_3 = (G \cap B_1) \cup Z_2$. We have $Y_3 \subseteq B_1$ and $a \vdash_1 Y_3 \vdash^0 G$. By repeated application of Lemma 55, we get

$$\begin{aligned} a \vdash_1 Y_3 \vdash^0 G \vdash a &\Rightarrow a \vdash_1 Y_3 \vdash a \\ &\Rightarrow \exists Y_4 \subseteq B_1.\, a \vdash_1 Y_3 \vdash_1 Y_4 \vdash^0 a \\ &\Rightarrow a \vdash_1 Y_3 \vdash_1 Y_4 \vdash_1 a \\ &\Rightarrow a \in Y_3. \end{aligned}$$

So, either $a \in G \cap B_1$, or $a \in Z_2$.

If $a \in G \cap B_1$, then $X \vdash^{k-1} a$. Since the induction step assumes $a \vdash^0 X$ already, this gives $X \dashv\vdash^{k-1} a$, and we apply the induction hypothesis to get $a \in X$.

If $a \in Z_2$, then

$$[a \vdash_1 G \cap B_1] \;\text{ and }\; [a \vdash_2 Z_2 \vdash_2 G \cap B_2] \;\text{ implies }\; a \vdash^0 G.$$

Therefore, $a \dashv\vdash^{k-1} G$ (we have $G \vdash^{k-1} a$ from the beginning of the induction step), which implies $a \in G$, by induction hypothesis. This means $X \dashv\vdash^{k-1} a$ and we apply the induction hypothesis again to get $a \in X$.

□

As a corollary of the previous two lemmas, we have

Theorem 58 *$\underline{B}$ is a irreducible information system. Moreover, both $\underline{B_1}$ and $\underline{B_2}$ embed into $\underline{B}$.*

In summary, we have proved the following.

Theorem 59 *The finite objects of the category* **Q** *has the amalgamation property. As a consequence of this, there exists a saturated irreducible information system, which in turn implies the existence of a saturated quasi-prime algebraic domain.*

12 A CPO of Irreducible Information Systems

One of the purposes of a universal domain is to ensure the existence of solutions to domain equations for denotational semantics of programming languages.

Another way to achieve the same goal is to introduce a cpo of irreducible information systems using a substructure relation. Various constructions on irreducible information systems can be shown to induce continuous functions on the cpo. The existence of the least fixed point for continuous functions then guarantees the existence of solution to domain equations. The advantage of this approach is that the solution is up to equality, rather than up to an isomorphism. We describe this approach now, which uses the idea first introduced in [11].

Definition 60 *Let* $\underline{A} = (A, Con_A, \vdash_A)$ *and* $\underline{B} = (B, Con_B, \vdash_B)$ *be irreducible information systems.* $\underline{A} \trianglelefteq \underline{B}$ *if*

1. $A \subseteq B$,
2. $X \in Con_A \iff X \subseteq A \;\&\; X \in Con_B$, *and*
3. $X \vdash_A a \iff X \cup \{a\} \subseteq A \;\&\; X \vdash_B a$.

When $\underline{A} \trianglelefteq \underline{B}$ we call $\underline{A}$ a subsystem of $\underline{B}$. Our definition of subsystem is exactly the same as that of Larsen and Winskel [11], except that it is restricted to irreducible systems. Note that for information system $\underline{A}$ and $\underline{B}$, if $A = B$ and $\underline{A} \trianglelefteq \underline{B}$, then $\underline{A} = \underline{B}$.

The relation $\trianglelefteq$ is a complete partial order on the *class* of irreducible information systems.

Theorem 61 *The relation* $\trianglelefteq$ *is a partial order with the least element*

$$\underline{\bot} = (\emptyset,\ \{\emptyset\},\ \emptyset).$$

If

$$\underline{A}_0 \trianglelefteq \underline{A}_1 \trianglelefteq \cdots \trianglelefteq \underline{A}_i \trianglelefteq \cdots$$

is an increasing chain of irreducible information systems, where

$$\underline{A}_i = (A_i, Con_i, \vdash_i),$$

then their least upper bound is

$$\bigsqcup_i \underline{A}_i = \left(\bigcup_i A_i,\ \bigcup_i Con_i,\ \bigcup_i \vdash_i \right).$$

PROOF. The only new component in the proof is to show that the least upper bound is irreducible. However, that follows directly from the fact that

the entailment relation is the cumulative union of that of the information systems in the chain.

□

Write **CPO** for the class of irreducible information systems under $\trianglelefteq$. **CPO** is not a cpo in the usual sense simply because they are not a set but a class. The subsystem relation $\trianglelefteq$ can be easily extended to $n-$tuples in the same way as described in [18]. A useful observation is that a unary operation F is continuous iff it is monotonic with respect to $\trianglelefteq$ and continuous on proposition sets, *i.e.*, for any $\omega-$chain

$$\underline{A}_1 \trianglelefteq \underline{A}_2 \cdots \trianglelefteq \underline{A}_i \trianglelefteq \cdots,$$

each proposition of $F\,(\bigcup_i \underline{A}_i\,)$ is a proposition of $\bigcup_i F\,(\,\underline{A}_i\,)$.

Many constructions can be introduced on irreducible information systems such as sum, product, lifting, function space, and even quasi-linear function space, as we have seen in Part II.

In the rest of the section we illustrate that function space $\rightarrow$ corresponds to a continuous operation

$$\rightarrow\colon\ \mathbf{CPO}^2 \rightarrow \mathbf{COP}.$$

Other constructions can be shown, in a similar way, to induce continuous operations on **CPO**.

Recall that the function space construction is given as follows.

$$\begin{aligned}
&\underline{A} \rightarrow \underline{B} = (Con_A \times B, Con_{A\rightarrow B}, \vdash_{A\rightarrow B}), \quad \text{where}\\
&\qquad X \in Con_{A\rightarrow B} \ \text{ iff } \ \bigcup \pi_A(X) \in Con_A \Rightarrow \pi_B(X) \in Con_B,\\
&\qquad X \vdash_{A\rightarrow B} (u,b) \ \text{iff } \ \{b' \mid \exists u'.u \vdash_A u' \ \&\ (u',b') \in X\} \vdash_B b
\end{aligned}$$

We have shown in Part II that $\rightarrow$ preserves irreducible information systems. $\rightarrow$ is monotonic in its first argument. Suppose $\underline{A} \trianglelefteq \underline{A}'$. Write

$$\underline{C} = (\,C,\, Con,\, \vdash\,) = [\underline{A} \rightarrow \underline{B}]$$

and

$$\underline{C}' = (\,C',\, Con',\, \vdash'\,) = [\underline{A}' \rightarrow \underline{B}].$$

Exactly as the case for information systems, we have $\underline{C} \trianglelefteq \underline{C}'$.

Let

$$\underline{A}_0 \trianglelefteq \underline{A}_1 \trianglelefteq \cdots \trianglelefteq \underline{A}_i \trianglelefteq \cdots$$

be a chain of irreducible information systems. Let (u, a) be a token of

$$[(\bigcup_i \underline{A}_i) \to \underline{B}].$$

Then clearly (u, a) is a token of $[\underline{A}_j \to \underline{B}]$ for some j, and thus is a token of $\bigcup_i [\underline{A}_i \to \underline{B}]$. From this we can deduce that $\to$ is continuous in its first argument. By a similar but easier proof we get that $\to$ is continuous in its second argument hence it is continuous.

As an application, we illustrate how to find a solution to the equation

$$X = X_\uparrow \to X$$

within irreducible information systems. Here $(\ \ )_\uparrow$ is the lifting construction specified as follows. Its use here makes sure that a non-trivial solution is obtained.

Let $\underline{A} = (A, Con, \vdash)$ be a irreducible information system. Define the *lift* of $\underline{A}$ to be $\underline{A}_\uparrow = (A', \vdash')$, where

- $A' = (\{0\} \times A) \cup \{0\}$,
- $X \vdash' Y \Leftrightarrow [0 \in Y \ \text{ or } \ \{c \mid (0, c) \in X\} \vdash_{\underline{A}} \{b \mid (0, b) \in Y\}]$.

It is easy to show that lifting preserves irreducible systems. It is an operation which, given a structure, produces a new one by joining a new token weaker than all the old ones. One can easily check that it gives a continuous operation.

Since the composition of continuous functions remains continuous, and any continuous function $F(x, y)$ of two variables gives rise to a continuous function $F(x, x)$ of one variable, the operation

$$X \longmapsto [X_\uparrow \to X]$$

is a continuous operation on irreducible information systems. It has a least fixed point

$$\underline{A} = \underline{A}_\uparrow \to \underline{A}.$$

This can be used as a model for the un-typed lambda calculus.

13 Concluding Remarks

We have introduced a new class of domains called quasi-prime algebraic domains. This class is a good candidate for the purpose of denotational semantics: it has a rich type structure. We have shown that quasi-prime algebraic

domains have many nice features, such as they form a cartesian closed category, a monoidal closed category, with appropriate morphisms. Moreover, there is a saturated quasi-prime algebraic domain and it is possible to solve domain equations in quasi-prime algebraic domains.

Irreducible information systems are an important technical tool which have helped us achieving all these. They not only have helped us resolving many technically difficult issues in an elegant way, but also have helped us presenting the results concisely. Although some of our results may look quite straightforward because of this, we think this is a virtue.

We would like to comment that quasi-prime algebraic domains are motivated from theoretical properties of certain constructions on domains. Perhaps a more appropriate way to motivate a new class of domains is to show adequacy or even full abstraction results for a proper programming language with a suitable operational semantics. Although the latter is probably more desirable, it is impossible or even not beneficial to have everybody follow the same line of thinking.

Clearly, the Plotkin powerdomain construction does not preserve quasi-prime algebraicity. However, the notion of quasi-primes makes sense beyond Scott domains. Our Theorem 8 is deliberately suggesting this. Categories of quasi-prime SFP domains, and quasi-prime L-domains may be introduced, for example.

The question of whether it makes sense to use stable functions as morphisms on quasi-prime algebraic domains has been settled, as a consequence of a recent result of the author [22]. The result implies that if both D and $[D \to_s D]$ (the stable function space) are Scott domains, then D must at least be prime algebraic.

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