Electronic Notes in Theoretical Computer Science 6 (1998)
URL: http://www.elsevier.nl/locate/entcs/volume6.html 13 pages

# Resolution in the Smyth Powerdomain

Guo-Qiang Zhang

*Department of Computer Science*
*University of Georgia*
*Athens, Georgia 30602, U.S.A.*

William C. Rounds

*Artificial Intelligence Laboratory*
*University of Michigan*
*Ann Arbor, Michigan 48109, U.S.A.*

E. N. T. C. S.

*Elsevier Science B. V.*

**Abstract**

The purposes of this paper are (i) to give a new, efficient representation of Smyth powerdomains as 0-inhabited information systems, following the style of Scott; (ii) to introduce, as a consequence of this representation, a generalized resolution method for an arbitrary Smyth powerdomain and show the soundness and completeness of this method. The connection of (i) with (ii) comes from the observation that the elements of a Smyth powerdomain are semantically equivalent to the basic notion of "clauses" as used in the area of logic programming.

## 1 Introduction

This paper proposes a generalized resolution rule for an arbitrary Smyth powerdomain, which we believe gives some insight into the foundations of proof systems for different kinds of logics. It began as a paper in *domain theory*, the mathematical foundation for *denotational semantics* of programming languages, developed, as is well-known, by Dana Scott and others. It works with the concept of *information systems*, Scott's method for representing abstract bounded complete partial orders (a.k.a. Scott domains) via a concrete presentation of their so-called *compact elements*.

Information systems play at least three important roles: pedagogical, technical, and foundational. The pedagogical role is what motivated Scott's original paper [8] – making domain theory more accessible. The technical role can be seen as a direct consequence of the former. By representing domains as information systems, one can work with more concrete and familiar notions of sets, relations, and containment. This is specially true for solving domain equations via information systems [11,13], as well as for determining the existence of universal objects [3,15]. The foundational role comes from the fact that essentially, information systems provide a method for effectively enumerating the compact elements of a domain.

Many ways to construct new Scott domains from old are by now known and well-understood. Among these are the *powerdomains*, of which one of the best known is Smyth's [9]. An interesting topic, pursued extensively by Winskel [12], Zhang [13–15,17] and others, is the representation of these constructions using information systems.

When it comes to the Smyth powerdomain, though, one notices that the straightforward way to derive an information system led to a very confusing proliferation of sets nested within other sets, which rather destroyed the original reason for introducing information systems in the first place. The present paper started as an attempt to rectify this phenomenon.

We have found a way to reduce this proliferation, using a new simplified Smyth powersystem construction, via the so called 0-*inhabited information systems*. (0-inhabited doesn't mean un-inhabited; just like having zero degree temperature doesn't mean no temperature.) These systems are similar to Scott's, with the non-essential difference being the presence of a special token 0, for "false", or inconsistency. This is the first main result; an interesting point of the paper, though, is not this result, but the idea which allows the simplification to work. Essentially, our construction is *guided* by the pattern of distributivity laws in general lattices. Our proofs, though, do not rely on the validity of these laws in the Smyth powerdomain, since we provide a direct proof that the powersystem construction works. The existence of such a simpler construction is perhaps not surprising, because the Smyth powerdomain on any domain is well-known to be distributive.

Here is how the distributivity laws have motivated the Smyth powersystem construction, as well as the proof of its correctness. One distributivity law, as stated in ([1], page 108), can be expressed in the form

$$\bigwedge_{i\in I} \bigvee A_i = \bigvee_{f:I \xrightarrow{\odot} \bigcup A_i} \bigwedge_{i\in I} f(i),$$

where we write $f : I \xrightarrow{\odot} \bigcup_{i\in I} A_i$ if $f(i)$ belongs to $A_i$ for every $i$ in $I$. (Such an $f$ is called a *choice function.*) What this shows is that (speaking very roughly) a conjunctive set of disjunctive sets can be represented as a disjunctive set of conjunctive sets. The (compact) elements of a Smyth powerdomain have exactly the form of a disjunctive set of conjunctive sets (we will call these or-and-sets), when their representation as conventional information systems is used. What we have exploited is the fact that the order-theoretic nature of the Smyth powersystem reflects in some sense the "logical content" of the distributive laws above.

Perhaps the most basic connection with resolution comes from the observation that in the 0-inhabited information system for the Smyth powerdomain, elementary pieces of information take exactly the form of a clause. We exploit this fact as well as the distributive laws in the second half of the paper, where we introduce a generalized resolution method for an arbitrary Smyth powerdomain, represented as a 0-inhabited information system. We provide a proof of the completeness of the generalized resolution rule. Even though it is for the completeness of generalized resolution, our proof seems to be rather much simpler than others for the propositional case, as it is derived from the order-theoretic nature of the Smyth powersystem via the notion of choice sets. It is interesting to note that a powerdomain construction, which might have been thought appropriate for describing disjunctive logic programming, is already present in the mechanics of the basic inference engine for *ordinary* logic programs. We hope that our insights can be extended to other concrete proof rules, perhaps most prominently things like SLD-resolution (see, e.g., Apt [2] or Lloyd [5]), as well as to resolution-based proof systems for other logics.

The paper consists of four sections. Section 2 reviews information systems and Scott domains, and present 0-inhabited information systems; Section 3 describes our powersystem construction, and Section 4 presents the generalized resolution method and proves its completeness, using order theory and the new powersystem construction.

## 2 Zero-inhabited information systems

We begin with the introduction of 0-inhabited information systems.

**Definition 2.1** *A* 0-*inhabited information system is a triple*

$$\underline{A} = (A, \vdash, 0),$$

*where $A$ is a countable set of tokens with $0 \in A$, and $\vdash$ is a relation between finite subsets of $A$ and elements of $A$. The relation $\vdash$ is required to satisfy the following conditions:*

*(i)* $\{a\} \vdash a$ *for any* $a \in A$,

*(ii)* $(X \vdash a \;\&\; X \subseteq Y) \Rightarrow Y \vdash a$,

*(iii)* $\{0\} \vdash a$ *for any* $a \in A$, *and*

*(iv)* $(X \vdash a \;\&\; Y, a \vdash b) \Rightarrow X, Y \vdash b$.

Our notation here follows that of the sequent calculus. So, $X, Y$ means $X \cup Y$, and $a, X$ means $\{a\} \cup X$. Note that Axiom 4, the cut rule, contains the special case of transitivity:

$$\{a\} \vdash b \;\&\; \{b\} \vdash c \Rightarrow \{a\} \vdash c.$$

A 0-inhabited information system is similar to an information system of Scott [8]. The difference is that instead of requiring the presence of "true", we require the presence of "false", denoted by 0. This is motivated by the fact that in resolution-based proof systems, one is interested in proving inconsistency, by deriving the empty clause. The empty clause has exactly the same status as the 0 for the Smyth powersystem, given in the next section.

We are particularly interested in the question of whether a set $X$ of tokens is inconsistent, or if the *deductive closure* $\overline{X}$ (see Definition 2.2) contains the special token 0. As a result, we don't need a special consistency predicate, as used in standard information systems; but that can be recovered from the axioms involving 0. The presence of this special token 0 makes the notion of inconsistency exactly the same as the "logical" one, in the sense that a set is inconsistent if and only if it "entails" all other tokens.

To recover the consistency predicate, let a finite set $X$ be consistent (written as $X \in \mathit{Con}$) if $X \not\vdash 0$. We need to check that $\mathit{Con}$ satisfies the relevant laws for ordinary information systems.

The first non-trivial law is

$$(X \subseteq Y \;\&\; Y \in \mathit{Con}) \Rightarrow X \in \mathit{Con},$$

which follows directly from Condition 2 in Definition 2.1. If $X \not\vdash 0$ and $X \vdash a$, then we must have $X, a \not\vdash 0$. This is because if $X, a \vdash 0$, then $X \not\vdash a$; for otherwise we can apply cut and deduce $X \vdash 0$, contradicting the assumption. We have, therefore,

$$(X \in \mathit{Con} \;\&\; X \vdash a) \Rightarrow X \cup \{a\} \in \mathit{Con},$$

another law for $\mathit{Con}$. Of course, we can no longer require all singleton sets $\{a\}$ to be consistent, as $\{0\}$ is not.

**Definition 2.2** *Suppose* $\underline{A} = (A, \vdash, 0)$ *is a* 0*-inhabited information system. An ideal element* $x$ *of* $\underline{A}$ *is just a deductively closed subset of* $A$*, i.e.*

$$(X \vdash a \;\&\; X \subseteq x) \Rightarrow a \in x.$$

*For any given set* $X$ *we write* $\overline{X}$ *for the deductive closure of* $X$*. We write* $|\underline{A}|$ *for the set of ideal elements of* $\underline{A}$*.*

Clearly, $\overline{\{0\}}$ is always the top element of the poset $(|\underline{A}|, \subseteq)$, and $\overline{\emptyset}$ is always the least element. (Care must be taken here, since the symbol for the empty set $\emptyset$ looks more like the zero element 0 than we wanted.) Of course, we have been a little bit liberal here by allowing *any* deductively closed set to be an ideal element. However, it is easy to switch back to the conventional definition if we need to, by requiring an ideal element not to include the token 0. To make a distinction, we can call an ideal element not including 0 a *consistent ideal element.*

To make the paper self-contained, we recall some basic domain-theoretic definitions.

**Definition 2.3** *A directed subset of a partial order* $(D, \sqsubseteq)$ *is a non-empty set* $X \subseteq D$ *such that for every* $x, y \in X$*, there is a* $z \in X$ *with* $x \sqsubseteq z \;\&\; y \sqsubseteq z$*. A complete partial order (cpo) is a partial order which has a bottom element and least upper bounds of directed sets. A subset* $X \subseteq D$ *is bounded (or* *compatible, consistent) if it has an upper bound in $D$. A compact (or finite, isolated) element $a$ of $D$ is one such that whenever $a \sqsubseteq \bigsqcup X$ with $X$ directed, we also have $a \sqsubseteq y$ for some $y \in X$. A cpo is algebraic if each element of it is the directed least upper bound of a set of compact elements. A cpo is $\omega$-algebraic if it is algebraic and the set of compact elements is countable. A Scott domain is an $\omega$-algebraic cpo in which every compatible subset has a least upper bound.*

The proof of the following representation theorem is almost the same as that for the earlier result of Scott [8], hence omitted.

**Theorem 2.4** *(Scott) A cpo $(D, \sqsubseteq)$ is a Scott domain if and only if there exists a 0-inhabited information system $\underline{A}$ such that $D$ is isomorphic to the set of consistent ideal elements of $\underline{A}$, under the order of set-inclusion.*

There is one small flaw here when a 0-inhabited information system does not contain any consistent tokens. We would have no consistent ideal elements. But this is not critical for us, because we can require that at least one consistent token to be present. Moreover, our results on resolution do not depend on it.

## 3 The Smyth powersystem

One standard definition of the Smyth powerdomain uses the technique called *ideal completion*. It can be found, for example, in Gunter [4] or Winskel [10], though the ideas in Smyth's original article are quite informative [9]. The details of ideal completion need not concern us here; it suffices to mention that the Smyth powerdomain of a Scott domain $D$ is the ideal completion of $(M[D], \sqsubseteq_\sharp)$, where $M[D]$ consists of the finite, non-empty, subsets of compact elements of $D$, and the preorder is defined as:

$$A \sqsubseteq_\sharp B \Leftrightarrow \forall b \in B \, \exists a \in A \; a \sqsubseteq_D b.$$

Although conceptually simple, powerdomains constructed from ideal completions have the disadvantage that they do not really live in the powerset $\mathcal{P}(D)$. The elements of $M[D]$ are already subsets of $D$, so the ideals are some sets of subsets of $D$, and are elements of $\mathcal{P}(\mathcal{P}(D))$ instead of $\mathcal{P}(D)$.

There exits a way to represent the Smyth powerdomain as a (subset of) the powerset $\mathcal{P}(D)$, though. Let $\uparrow X := \{y \mid \exists x \in X \; y \sqsupseteq x\}$. We have $\uparrow B \subseteq \uparrow A$ if and only if $A \sqsubseteq_\sharp B$. The Smyth powerdomain of a Scott domain $D$ consists of, as the finite elements, all subsets $\uparrow A$ with $A$ a non-null finite set of compact elements of $D$. Such a collection of elements forms a partial order under the superset ordering, and we have to add to it the limits (least upper bounds) to get a cpo. The limit point of an $\omega$-increasing chain

$$\uparrow A_0 \supseteq \uparrow A_1 \cdots \supseteq \uparrow A_i \supseteq \cdots$$

is the (non-empty) intersection

$$\bigcap_{i \in \omega} \uparrow A_i.$$

Let $\mathcal{P}_S(D)$ be the set consisting of $\uparrow A$ with $A \in M[D]$, together with the limits of the $\omega$-increasing chains of such elements.

These elements turn out to be exactly the *compact, saturated* sets of $D$ (see [1]). Here, a subset of $D$ is called *compact* if it is compact in the standard topological sense, with respect to the Scott topology on $D$; a subset of $D$ is called *saturated* if it is the intersection of Scott open sets containing it. The following well-known result (see, e.g., Abramsky and Jung [1] or Zhang [17]) gives a way to make the elements of the Smyth powerdomain stay within the powerset $\mathcal{P}(D)$.

**Theorem 3.1** *Let $D$ be a Scott domain. Then $(\mathcal{P}_S(D), \sqsubseteq_\sharp)$ is the Smyth powerdomain of $D$: it is an $\omega$-algebraic cpo isomorphic to the ideal completion of the preorder $(M[D], \sqsubseteq_\sharp)$.*

Note that the bottom element of the Smyth powerdomain is $D$, which is the upward closure $\uparrow\perp_D$. We also need to remark here on the presence of the empty set: it does not belong to $\mathcal{P}_S(D)$, although one can easily adjoin it, whence it becomes the "most informative" but "inconsistent" element.

We now introduce a Smyth power-construction on 0-inhabited information systems. Given a 0-inhabited information system $\underline{A}$, how do we construct a 0-inhabited information system for the Smyth powerdomain? It seems that the answer should be simple, given that the Smyth powerdomain construction is well-understood. However, it is surprisingly non-trivial to do this in the simplest way. The reason is that when representing a Scott domain by an information system, the compact elements of the domain are in one-to-one correspondence with (the deductive closure of consistent) finite "and-sets" of tokens. Since the Smyth powerdomain is built from finite "or-sets" of compact elements of a base domain, it is natural that a *single* token for the Smyth powersystem should be already a finite or-set of finite and-sets of tokens from the base system. Then the compact elements of the powersystem would be represented by finite and-sets of these tokens, a three-level nesting (and-or-and-sets). We would like to reduce this by one level using (and-or-sets only).

In our definition of the powersystem (see later), we will use simplified tokens, merely finite or-sets of tokens from the original system (considered as disjunctive information), which is reasonable from the powerset point of view.

To understand why this simplification is possible, we consider an intuitive example. Suppose in the original information system we have tokens $t_1, t_2, a, b, c, d$, with $t_1$ such that $\{t_1\} \vdash a$, $\{t_1\} \vdash b$ and $\{a, b\} \vdash t_1$. This means $t_1$ and $a \wedge b$ are "logically" equivalent. Similarly, let $t_2$ be logically equivalent to $c \wedge d$. As far as domains are concerned, this shows that elements such as $t_1$ and $t_2$ are redundant. However, with the presence of $t_1$ and $t_2$, the or-set $\{t_1, t_2\}$ represents a finite (or-)set of compact elements of the base domain. Without

$t_1$ and $t_2$, the powerdomain element $\{t_1, t_2\}$ is not directly representable as a token in the simpler powersystem – we would need to use the or-and-set $\{\{a, b\}, \{c, d\}\}$ instead. So one idea to keep things simple at the powersystem level is to have enough redundant tokens in the base system, making it "conjunctively closed" (indeed, this approach is taken earlier in Zhang [14]; we also use this idea in the next proof). But this seems paradoxical: in order to keep things simple in the powersystem, we have to create redundant tokens in the base system.

The solution to this impasse involves a simple idea in the distributive laws of propositional logic. The or-set $\{t_1, t_2\}$ can be captured in an indirect way without using redundant tokens $t_1, t_2$. We can think of $\{t_1, t_2\}$ as the proposition $t_1 \vee t_2$, which is equivalent to $(a \wedge b) \vee (c \wedge d)$. In propositional logic, every proposition is equivalent to a conjunctive normal form. In particular, we have

$$(a \wedge b) \vee (c \wedge d) \Leftrightarrow (a \vee c) \wedge (a \vee d) \wedge (b \vee c) \wedge (b \vee d).$$

This *suggests* (but it does not validate) that we may be able to capture the informational content of the or-set $\{t_1, t_2\}$ by the following four simple tokens (which are or-sets) in the powersystem: $\{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}$. This is the key idea which makes our simplified Smyth powersystem work.

**Remark**. In the remaining of the paper, we will talk about consistency and entailment on an arbitrary set of tokens, by enforcing compactness. So, a set $w \subseteq A$ is consistent with respect to $(A, \vdash_A, 0_A)$ if for every finite subset $w'$ of $w$, we have $w' \not\vdash_A 0_A$. Similarly, $w \vdash_A a$ for infinite $w$ if there exists a finite subset $w'$ of $w$ such that $w' \vdash_A a$.

**Definition 3.2** *Let $\underline{A}$ be an information system. The Smyth powersystem of $\underline{A}$ is the structure $\mathcal{P}\underline{A} = (C, \vdash, 0)$ where*

$$C = \{u \mid u \textit{ is a finite (or-)subset of tokens in } A\}$$

$$X \vdash u \textit{ iff for any consistent set } w \in C \ (\textit{w.r.t. } \underline{A}),$$

$$[\forall \alpha \in X.\, w \cap \alpha \neq \emptyset] \Rightarrow \exists a \in u.\ w \vdash_A a.$$

In fact, we can take $\emptyset$ as the 0 token of the powersystem – note that we do not require the 0 token to be unique.

A set $w$ with the property

$$[\forall \alpha \in X.\, w \cap \alpha \neq \emptyset]$$

is called a *choice set* [17] of $X$. Choice sets do not have to be consistent, although consistent choice sets are most important here.

It should be helpful to point out that a (and-or-)set $X$ is consistent with respect to $\mathcal{P}\underline{A}$ (i.e., $X \not\vdash \emptyset$) if and only if there exists a consistent (with respect to $\underline{A}$) choice set for $X$.

When $X = \{u\}$, we simplify the notation by writing $u \vdash v$ for $\{u\} \vdash v$. In this case any singleton subset of $u$ is a choice set of $\{u\}$, so $u \vdash v$ if and only if for each $a$ in $u$, there exists a $b$ in $v$, such that $\{a\} \vdash_A b$. This suggests that, indeed, the powersystem $\mathcal{P}\underline{A}$ should capture the Smyth ordering.

We now can state the main result of this section.

**Theorem 3.3** *The collection of consistent ideal elements of the Smyth powersystem $\mathcal{P}\underline{A}$ is isomorphic to the Smyth powerdomain of $(|\underline{A}|, \subseteq)$.*

**Proof (Sketch)** Instead of proving directly that $\mathcal{P}\underline{A}$ and the Smyth powerdomain of $(|\underline{A}|, \subseteq)$ are isomorphic, we prove the following:

(i) $(|\underline{\hat{A}}|, \subseteq)$ is isomorphic to $(|\underline{A}|, \subseteq)$, where $\underline{\hat{A}}$ is the *conjunctive closure* of $\underline{A}$, in the sense that $\underline{\hat{A}}$ augments $\underline{A}$ by introducing a new token $a$ for each consistent set $X$ of $\underline{A}$ in such a way that

$$X \vdash a \text{ and } \forall b \in X\ \{a\} \vdash b$$

in the new system $\underline{\hat{A}}$.

(ii) $(|\mathcal{P}\underline{\hat{A}}|, \subseteq)$ is isomorphic to $(|\mathcal{P}\underline{A}|, \subseteq)$.

(iii) $(|\mathcal{P}\underline{\hat{A}}|, \subseteq)$ is isomorphic to the Smyth powerdomain of $(|\underline{\hat{A}}|, \subseteq)$.

It is straightforward to prove (i) .

For (ii), we show that if $a$ is a member of $u$ (a token of the powersystem $\mathcal{P}\underline{\hat{A}}$) introduced to $\hat{A}$ to represent a consistent set $X$ of $\underline{A}$, then we can find a finite set of tokens of $\mathcal{P}\underline{\hat{A}}$ that is "equivalent" to $u$, without using $a$. We can take this (and-)set to be

$$Z := \{\{b\} \cup (u \setminus \{a\}) \mid b \in X\}.$$

Suppose $w$ is a consistent choice set of $\{u\}$. If $a \notin w$, then clearly $w \vdash_{\hat{A}} c$ for some $c \in (u \setminus \{a\})$. If $a \in w$, then $w \vdash b$ for each $b \in X$. This shows that we have $\{u\} \vdash \{b\} \cup (u \setminus \{a\})$ for each $b \in X$ with respect to $\mathcal{P}\underline{\hat{A}}$. On the other hand, if $w$ is a consistent choice set of $Z$, then $w$ intersects each member of $Z$. It follows that either $w$ intersects some member of $u$, or else $w$ contains $X$. In both cases we have $Z \vdash u$ with respect to $\mathcal{P}\underline{\hat{A}}$. In summary, we have shown that the new tokens in $\underline{\hat{A}}$ do not change the Smyth powerdomain structure.

To prove (iii), note that with the help of new tokens in $\underline{\hat{A}}$, every compact element of the Smyth powerdomain of $(|\underline{\hat{A}}|, \subseteq)$ can be captured by a finite and-set $Y$, each element of which represents a compact element of $(|\underline{\hat{A}}|, \subseteq)$. However, although $\mathcal{P}\underline{\hat{A}}$ may not be conjunctively closed, elements determined by sets such as $Y$ are in one-to-one correspondence to and-or-sets of the form

$$\{\{b\} \mid b \in Y\},$$

where each $\{b\}$ is regarded as a single token (an or-set, if you wish) of $\mathcal{P}\underline{\hat{A}}$.

In addition, each finite, consistent and-or-set $\{u_i \mid i \in I\}$ for the powersystem $\mathcal{P}\underline{\hat{A}}$ represents the same element as what is determined by the set consists of $a$'s such that $a$ is a representative of a consistent (and-)set (w.r.t. $\underline{A}$) of the form

$$\{f(i) \mid f : I \xrightarrow{\odot} \bigcup_{i \in I} u_i \text{ is a choice function}\}.$$

□

It can be seen that in the proof of item (ii) above, we were guided by the simpler version of the distributive law:

$$(\bigwedge X) \vee \bigvee Y = \bigwedge_{b \in X} (b \vee \bigvee Y)$$

when eliminating a single element in $\hat{A} \setminus A$.

Detailed steps for the validity of the statement given at the end of the proof for (iii) can be supplied by closely following the pattern of general distributive laws.

## 4 Generalized resolution

A token in $C$ for the Smyth powersystem (see Definition 3.2) can be thought of as a clause. In other words, if $u = \{a_1, a_2, \ldots, a_k\}$ is a member of the powersystem, one should think of it as the disjunction

$$a_1 \vee a_2 \vee \ldots \vee a_k.$$

A finite set of such tokens, $\{u_1, u_2, \ldots, u_n\}$, can be properly regarded as a *clause set*, as in *logic programming* [2,5]. Since every propositional formula is logically equivalent to a clause set, the Smyth powerdomain carries an interesting logical content, as well as an order-theoretic one.

We would like to see now how the order-theoretic content of the Smyth powerdomain gives rise to a sound and complete generalized resolution method. We first give a brief account of the standard propositional resolution in light of the powersystem construction.

Fix now a specific base 0-inhabited information system

$$\underline{\mathsf{B}} = (\mathsf{B}, \vdash, 0),$$

for propositional logic. The token set $\mathsf{B}$ is

$$\{0, x_1, \neg x_1, x_2, \neg x_2, \ldots, x_k, \neg x_k, \ldots\}.$$

The entailment relation $\vdash$ is the smallest relation such that

$$X \vdash 0 \;\text{ if }\; \{x_k, \neg x_k\} \subseteq X$$

for some $k \geq 1$. One should think of $\mathsf{B}$ as the set of propositional variables $x_k$ and their negation $\neg x_k$. (This is a good example showing how clumsy it would be to require the system be conjunctively closed.)

In the powersystem $\mathcal{P}\mathsf{B}$, the intuitive way to think about a choice set $w$ is a (partial) truth assignment which assigns all the literals in $w$ to true, and all other literals to "undefined". Of course, this makes sense only if $w$ is consistent, *i.e.*, it does not contain both a literal and its negation. Note that for tokens $u$, $v$ in the powersystem, $u \vdash v$ if and only if $u \subseteq v$. This is because the base system is flat, in the sense that $a \vdash_{\mathsf{B}} b$ if and only if $a = b$.

Finite and-or-sets of $\mathsf{B}$ can be used to represent general boolean formulas over $\mathsf{B}$. For example, the and-or-set

$$\{\{x_1, x_2\}, \{\neg x_2, x_3\}, \{\neg x_1\}, \{x_1, x_3\}\}$$

represents the following boolean formula in conjunctive normal form:

$$(x_1 \vee x_2) \wedge (\neg x_2 \vee x_3) \wedge \neg x_1 \wedge (x_1 \vee x_3).$$

The resolution principle [6] (see also [2,5,16]) deals with the problem of demonstrating the validity of a proposition by detecting the contradiction of its negation. Instead of trying to transform the original proposition into an equivalent tautology, the resolution method starts with a proposition in conjunctive normal form (or a set of clauses) and then repeatedly applies the resolution rule, the major advantage being that only one rule is needed. When applied to two resolvable clauses, one containing the literal $x$ and the other containing $\neg x$, the resolution rule produces a resolvent, which is a clause consisting of the union of the two clauses with $x$ and $\neg x$ removed.

Formally, the resolution rule is given as

$$\frac{u \cup \{\, x \,\} \qquad v \cup \{\, \neg x \,\}}{u \cup v},$$

with $u$, $v$ clauses and $x$ a literal. The soundness of this rule says that the clause $u \cup v$, called the *resolvent* , is a logical consequence of $u \cup \{\, x \,\}$ and $v \cup \{\, \neg x \,\}$. Restated with respect to the Smyth powersystem, soundness means that if $u \cup \{\, x \,\}$ and $v \cup \{\, \neg x \,\}$ are both valid at a truth assignment as provided by a consistent choice set $w$, then so does $u \cup v$. This is because either $x \notin w$, or $\neg x \notin w$, which implies $w \cap u \neq \emptyset$, or $w \cap v \neq \emptyset$. Therefore, $w \cap (u \cup v) \neq \emptyset$. This way, the semantic content of resolution is captured in the Smyth powersystem $\mathcal{P}\mathsf{B}$. As a special instance, we have

$$\{\{x_k\}, \{\neg x_k\}\} \vdash \emptyset,$$

as well as $\{\emptyset\} \vdash u$ for any $u$. As remarked earlier, the empty set has the status of the zero element in any Smyth powersystem.

The Smyth powersystem (Definition 3.2) allows a rather straightforward generalization of the resolution method mentioned above. For an arbitrary powersystem $\mathcal{P}\underline{A}$, we have the following *generalized resolution* rule, first introduced for first order logic by Robinson [7]:

$$\frac{u_1 \cup \{a_1\} \quad u_2 \cup \{a_2\} \quad \cdots \quad u_k \cup \{a_k\}}{u_1 \cup u_2 \cup \cdots \cup u_k}, \qquad \text{where } \{a_i \mid 1 \leq i \leq k\} \vdash_A 0_A.$$

We call $u_1 \cup u_2 \cup \cdots \cup u_k$ a *resolvent* of $u_1 \cup \{a_1\}, u_2 \cup \{a_2\}, \cdots, u_k \cup \{a_k\}$.

To check if a set $X$ of or-sets in $\mathcal{P}\underline{A}$ is consistent, we repeatedly apply the above rule to an arbitrary subset of $X$ and add the resolvent to $X$. We then check to see if the empty set is present. If yes, then $X$ is inconsistent. If no empty set can be found in this process, then $X$ is consistent. For finite $X$, this process terminates because all resolvents must be subsets of $\bigcup X$.

**Definition 4.1** *Let $X$ be an arbitrary set of tokens in a Smyth powersystem $\mathcal{P}\underline{A}$. We write $\mathsf{R}(X)$ for the least set $Y$ such that*

*(i) $Y \supseteq X$, and*

*(ii) if $u$ is the resolvent of a finite subset of $Y$, then $u$ belongs to $Y$.*

Of course, we want to make sure that if $\emptyset \in \mathsf{R}(X)$, then $X \vdash \emptyset$ and so it is indeed inconsistent (according to Definition 3.2). Conversely, if $\emptyset \notin \mathsf{R}(X)$, then $X$ must be consistent – there exists a consistent (w.r.t $\underline{A}$) choice set for $X$. This is exactly what we call the soundness and completeness of the generalized resolution method.

It is easy to see that the method is sound: if $u$ is a resolvent of a finite subset $Y$ of $X$, then for any consistent choice set $w$ of $X$, $w$ intersects each member of $Y$. However, $w$ cannot intersect each member of $Y$ exactly at the $a_i$s (see the generalized resolution rule), since it is consistent. Therefore, $u \cap w$ must be non-empty. This shows $X \vdash u$. Similarly, for any $u \in \mathsf{R}(X)$, we have $X \vdash u$.

The more interesting part is the completeness.

**Theorem 4.2** ***(Completeness of generalized resolution)*** *Let $X$ be an arbitrary set of tokens in the Smyth powersystem $\mathcal{P}\mathsf{B}$. Then $X$ is inconsistent (i.e., $X \vdash \emptyset$) only if $\emptyset \in \mathsf{R}(X)$.*

The proof proceeds with two lemmas.

**Lemma 4.3** *Let $X$ be an arbitrary set (finite or infinite) of tokens in a Smyth powersystem. Then every choice set $w$ for $X$ contains a minimal choice set $m$ with respect to set-inclusion.*

**Proof.** We first observe that $X$ has a choice set if and only if it does not contain the special token $\emptyset$. Fix a choice set $w$ for $X$. Consider the subset ordering defined on the set of choice sets for $X$ which are subsets of $w$. Given any chain $C$ of such choice sets of $X$, the intersection $\bigcap C$ is again a choice set. In fact, for any token $u$ in $X$, $u \cap \bigcap C$ must be non-empty because otherwise for each $a$ in $u$, we can find a member $m(a)$ in $C$ for which $a$ is not a member. However, $C$ is a chain, and this means there is some $m(a_0)$ which is the least member in the *finite* (since $u$ is finite) collection of choice sets

$$\{m(a) \mid a \in u \;\&\; m(a) \in C \;\&\; a \notin m(a)\}.$$

The intersection of $u$ and $m(a_0)$ would be empty. This is a contradiction, since $m(a_0)$ is assumed to be a choice set of $X$. This contradiction shows that $u \cap \bigcap C \neq \emptyset$ for any $u \in X$, and hence $\bigcap C$ is again a choice set for $X$.

Since any chain of choice sets has a greatest lower bound as provided by the intersection, there is a minimal choice set $m$ below $w$, by Zorn's Lemma. □

**Lemma 4.4** *A choice set $m$ for $X$ is minimal if and only if for each $a$ in $m$, there exists some $u$ in $X$, such that $m \cap u = \{a\}$.*

**Proof.** Clearly, a choice set must be minimal if it has the property that for each $a$ in $m$, there exists a $u$ in $X$ such that $m \cap u = \{a\}$. On the other hand, a minimal choice set must have this property. Otherwise, there exists some $b$ in $m$ such that for each $v$ in $X$, $m \cap v \neq \{b\}$. But $m \cap v$ must be non-empty; so it contains some member other than $b$. It can be easily seen that $m \setminus \{b\}$ would still be a choice set, contradicting the minimality of $m$. □

**Proof of Theorem 4.2** Suppose $X \vdash \emptyset$. We need to show that $\emptyset \in \mathsf{R}(X)$. For the purpose of deriving a contradiction, let's assume that $\emptyset$ does not belong to $\mathsf{R}(X)$. The set $\bigcup \mathsf{R}(X)$ is a choice set for $\mathsf{R}(X)$. By Lemma 4.3, $\mathsf{R}(X)$ has a minimal choice set $m$. It suffices to prove that $m$ is consistent, with respect to $\underline{A}$. By Lemma 4.4, the minimal choice set $m$ has the property that for each $a \in m$, there is a $u \in \mathsf{R}(X)$ such that $u \cap m = \{a\}$. If $m$ is inconsistent, then there exists a finite subset $m'$ of $m$ such that $m' \vdash_A 0_A$. For each $a \in m'$, fix a $u_a \in \mathsf{R}(X)$ such that $u_a \cap m' = \{a\}$. Now consider the finite set

$$Y := \{u_a \mid a \in m'\}.$$

We can apply the generalized resolution rule to $Y$ to derive the resolvent

$$\bigcup \{u_a \setminus \{a\} \mid u_a \in Y \ \&\ u_a \cap m' = \{a\}\},$$

which should remain in $\mathsf{R}(X)$. However, this resolvent is no longer overlapping with $m$, contradicting the assumption that $m$ is a choice set of $\mathsf{R}(X)$. □

## 5 Concluding remarks

We have introduced a concise presentation of the Smyth powerdomain based on the notions of choice set and 0-inhabited information systems. This result leads to a sound and complete generalized resolution method. The proof for the completeness of resolution is interesting in itself, because it is an order-theoretic one. Moreover, the proof can be used directly for the completeness of the standard resolution for propositional logic–the standard proofs based on mathemtiacal induction are ugly and not informative!

A potential further benefit of the generalized resolution is that we are no longer constrained by the limitations of propositional logic. If we can derive a general clause normal form for any logic, then the generalized resolution rule should provide a method for deciding inconsistency in a straightforward way.

## References

[1] S. Abramsky and A. Jung, Domain theory, in: *Handbook of Logic in Computer*

*Science*, Vol 3, (Clarendon Press, 1995).

[2] K. R. Apt, Logic programming, in: *Handbook of Theoretical Computer Science*, Vol. B (MIT Press, 1990 493–574).

[3] M. Droste and R. Göbel, Universal information systems, *Internat. J. Foundations Comput. Sci.* vol 1: 413–424, 1990.

[4] C. Gunter, *Semantics of Programming Languages – Structures and Techniques*, Foundations of Computing (MIT Press, 1992).

[5] J. Lloyd, *Foundations of Logic Programming*, 2nd extended ed., (Springer-Verlag, New York, 1987).

[6] J.A. Robinson, A machine-oriented logic based on the resolution principle, JACM, 12, (1) 1965.

[7] J.A. Robinson, The generalized resolution principle. *Machine Intelligence*, vol 3, 77–93, Edinburgh University Press, 1968.

[8] D. S. Scott, Domains for denotational semantics, in: *Lecture Notes in Computer Science* **140** (1982) 577–613.

[9] M. Smyth, Powerdomains, *Journal of Computer and Systems Science*, 16: 23–36, 1977.

[10] G. Winskel, On powerdomains and modality, *Theoretical Computer Science* 36:127–137, 1985.

[11] G. Winskel and K. Larsen, Using information systems to solve recursive domain equations effectively, *Lecture Notes in Computer Science* vol 173, 1984.

[12] G. Winskel, *The Formal Semantics of Programming Languages*, MIT Press, 1993.

[13] G.-Q. Zhang, dI-domains as prime information systems, *Information and Computation*, 100:151–177, 1992.

[14] G.-Q. Zhang, A representation of SFP, *Information and Computation*, 110:233–263, 1994.

[15] G.-Q. Zhang, Quasi-prime algebraic domains, *Theoretical Computer Science*, 155:221–264, 1996.

[16] G.-Q. Zhang, The Davis-Putnam resolution procedure, in: *Advances in Logic Programming and Automated Reasoning*, Vol 2, Chapter 4, (Ablex Publishing Corporation, 1995).

[17] G.-Q. Zhang, *Logic of Domains*, Progress in Theoretical Computer Science (Birkhauser Boston, Inc., Boston, MA, 1991).