Stable Neighbourhoods

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Abstract

This paper studies topological aspects of stable functions. A basic notion of open sets called stable neighbourhoods is introduced. Stable neighbourhoods are shown to characterize stable functions. Constructions of stable neighbourhoods in the category of coherent spaces and the category of dI-domains are given with respect to type constructions in these categories.

1 Introduction

A less standard but important framework for the denotational semantics of programming languages is the category of dI-domains with stable functions. Stable functions were discovered by Berry [2] from the study of the full-abstraction problem for typed λ-calculi. He introduced the category of dI-domains which are special kinds of Scott domains with a more operational nature. The functions on dI-domains are stable functions under an order which takes into account the manner in which they compute. Stable functions have also found use in modeling concurrency. The stable event structures of Winskel [9] gives a category in which constructions like the partially synchronous product can be used to model languages like CCS and CSP quite smoothly. Another important category which uses stable functions is the coherent spaces brought into popularity by Girard. They have been recently used to model system F [5], and for a semantics of linear logic [6].

Topological concepts have been shown very useful to computer science. We can think of a topological space as a data type, open sets as properties, and points in the space as computations. This idea has been shown to be particularly useful in understanding the relationship between denotational semantics and program logics ([1], [11]).

A claimed advantage of dI-domains (with stable functions) over Scott domains seems to be that 'finiteness is finiteness' ([6]), that is, in dI-domains finite elements are exactly those which dominate finitely many elements. However I think it is fair to point out that one should not over stress one aspect (finiteness) and forget the other (topological treatment). There is an elegant topological characterization of continuous functions on Scott domains which makes it convenient to study the relationship of denotational semantics and program logic. How about stable functions?

The purpose of this paper is to explore the topological properties related to stable functions and study the constructions on the resulting topological spaces. We introduce stable neighbourhoods, to characterize stable functions. A dI-domain can be seen as a collection of computations of certain type. The stable neighbourhoods of the dI-domain can be taken as properties about the computations. Constructions on dI-domains can be seen as ways to combine computations together. The corresponding constructions on stable neighbourhoods generate proof rules to reason about computation in the stable world. As many key rules for Scott's framework of continuous functions are no longer sound with respect to the stable order (Note that the rule

$$\frac{A \vdash A' \qquad B' \vdash B}{A' \to B' \vdash A \to B},$$

typical for Scott's function space, is not appropriate for stable functions, as will be seen later), we have to investigate the constructions of stable neighbourhoods carefully to find the right proof rules.

It is regarded as important to be able to determine the points (computations) by their properties. And the properties of a computation should be determined by those of its components. The constructions of stable neighbourhoods should meet these requirements, and that makes them non-trivial.

The organization of the paper is as follows. Section 2 introduces stable neighbourhoods and studies their basic properties. Section 3 deals with constructions of stable neighbourhoods in the category of dI-domains. Section 4 deals with constructions of stable neighbourhoods with respect to type constructions in the category of coherent spaces.

2 Stable Neighbourhoods

Scott open sets play an important role in domain theory. They are essential to the logics of domains where open sets are regarded as properties, and constructions on them provide proof rules. This is because Scott open sets have many nice properties, which include

• a function is continuous if and only if the inverse image of an open set is open;

• for continuous functions f and g, $f \sqsubseteq g$ if and only if $f^{-1}(O) \subseteq g^{-1}(O)$ for all open set O;

• Scott open sets form a topology.

For the dI-domains introduced by Berry, is there a similar notion of 'open sets' which characterize stable functions?

To discuss this question, let us recall briefly Berry's category of dI-domains and stable functions. A *dI-domain* is a Scott domain D which satisfies

- axiom d: $\forall x, y, z \in D.\ y \uparrow z \Longrightarrow x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$
- axiom I: $\forall d \in D^0.\ \mid \{ x \mid x \sqsubseteq d \} \mid < \infty$

A function $f : D \to E$ between dI-domains D and E is *stable* if it is Scott-continuous and preserves meets of pairs of compatible elements, i.e.,

$$\forall x,\ y \in D.\ \ x \uparrow y \Longrightarrow f(x \sqcap y) = f(x) \sqcap f(y).$$

Let $f,\ g$ be stable functions from D to E. Say f stably less than g, written $f \sqsubseteq_s g$, if $\forall x,\ y \in D.\ \ x \sqsubseteq y \Longrightarrow f(x) = f(y) \sqcap g(x)$. Write $[D \to_s E]$ for the set of stable functions from D to E. DI-domains with stable functions form a cartesian closed category, by a theorem of Berry [2]. It implies that there is a rich type structure in this category.

Consider the stable functions from $\mathcal{O} \times \mathcal{O}$ (the Cartesian product with the coordinatewise order) to $\mathcal{O}$, where $\mathcal{O}$ is a dI-domain consisting of two elements $\perp$ and $\top$, with $\perp \sqsubseteq \top$. Suppose there were such a topology. Then the topology on $\mathcal{O}$ must contain $\{\top\}$ as an open set: otherwise we may get non-monotonic functions. The inverse image of the stable function

$$(\top\ ,\ \perp) \longmapsto \top, \quad (\perp\ ,\ \top) \longmapsto \perp$$

on $\{\top\}$ is $\{(\top\ ,\ \top)\ ,\ (\top\ ,\ \perp)\}$ and the inverse image of the stable function

$$(\perp\ ,\ \top) \longmapsto \top, \quad (\top\ ,\ \perp) \longmapsto \perp$$

on $\{\top\}$ is $\{(\top\ ,\ \top)\ ,\ (\perp\ ,\ \top)\}$. Hence their union $\{(\top\ ,\ \top)\ ,\ (\perp\ ,\ \top)\ ,\ (\top\ ,\ \perp)\}$ would be again an open set. That means the topology on $\mathcal{O} \times \mathcal{O}$ coincides with the Scott topology, which allows the non-stable 'parallel-or':

$$(\perp\ ,\ \top) \longmapsto \top$$

$$(\top\ ,\ \perp) \longmapsto \top$$

$$(\perp\ ,\ \perp) \longmapsto \perp$$

Therefore the required topology does not exist.

It is still meaningful, however, to ask whether there is any class of sets which characterize stable functions in the following sense:

- a function is stable if and only if the inverse image of a set in this class is still a set in the class;
- for stable functions f, g, $f \sqsubseteq_s g$ if and only if $f^{-1}(O) \sqsubseteq g^{-1}(O)$ for all set O in this class, where $\sqsubseteq$ is a suitable order.

The answer is *yes*.

Let D be a Scott domain. We have $[D \to \mathcal{O}] \cong \Omega(D)$, *i.e.*, the continuous functions $[D \to \mathcal{O}]$ with the pointwise order are isomorphic to Scott open sets of D, with set inclusion as the order. The isomorphism is given by $f \longmapsto f^{-1}(\top)$. Now consider a dI-domain D and a stable function $f : D \to \mathcal{O}$. $f^{-1}\{\top\}$ is a Scott-open set as f is continuous. If $x \uparrow y$ and $x, y \in f^{-1}\{\top\}$ then $x \sqcap y \in f^{-1}\{\top\}$ since stable functions preserve meets of compatible elements. This simple analysis leads to

Definition 2.1 *Let D be a dI-domain. U is a stable neighbourhood of D if*

- *U is Scott-open and*
- $(x \uparrow y \;\&\; x, y \in U) \Longrightarrow x \sqcap y \in U.$

Write the set of stable neighbourhoods of a dI-domain D as $\mathbf{SN}(D)$. $\mathbf{SN}(D)$ does not necessarily form a topology. It is closed under finite intersections but not arbitrary unions. However, stable neighbourhoods are closed under disjoint unions.

Definition 2.2 *A disjunctive space is a pair $(S, \mathcal{N})$ where S is a set and $\mathcal{N}$ is a collection of subsets of S satisfying the following properties:*

- $\mathcal{N}$ is closed under finite intersection
- $\mathcal{N}$ is closed under disjoint union, i.e.
  $(\forall i \in I.\, U_i \in \mathcal{N}) \;\&\; (U_i \cap U_j = \emptyset$ for every $i \neq j) \Longrightarrow \bigcup_{i \in I} U_i \in \mathcal{N}$

A subset in $\mathcal{N}$ is called a neighbourhood.

Clearly a topological space is a disjunctive space, but not *vice versa*. However, many notions on topological spaces can be introduced to disjunctive spaces without much change. A *base* of the disjunctive space $\mathcal{N}$ is a subset $\mathcal{B} \subseteq \mathcal{N}$ such that every neighbourhood is the disjoint union of members from $\mathcal{B}$. A *sub-base* of $\mathcal{N}$ is a subset $\mathcal{A} \subseteq \mathcal{N}$ such that every neighbourhood is the disjoint union of finite intersections of elements of $\mathcal{A}$.

Suppose $x \in S$ and $\mathcal{N}$ is a disjunctive space on S. Let $\mathcal{P}_x$ be the collection of neighbourhoods each of which contains x. Then $\mathcal{P}_x$ has the following properties:

- $U \in \mathcal{P}_x \;\&\; U \subseteq V \in \mathcal{N} \implies V \in \mathcal{P}_x$;
- $U \in \mathcal{P}_x \;\&\; V \in \mathcal{P}_x \implies U \cap V \in \mathcal{P}_x$;
- for any disjoint collection of U_i's in $\mathcal{N}$, $\bigcup_{i \in I} U_i \in \mathcal{P}_x \Longrightarrow U_i \in \mathcal{P}_x$ for some i.

A space is called T_0 if whenever $\mathcal{P}_x = \mathcal{P}_y$, we also have $x = y$. Forgetting about the point x, we call any collection $\mathcal{P}$ of neighbourhoods with the foregoing three properties a *complete prime filter* in $\mathcal{N}$. A space is *sober* if it is T_0 and for each complete prime filter $\mathcal{P}$ in $\mathcal{N}$, there is some point x such that $\mathcal{P}_x = \mathcal{P}$. Intuitively a space is sober if it is completely determined by its lattice of properties (neighbourhoods).

The collection of stable neighbourhoods of a dI-domain indeed forms a disjunctive space. Moreover, $(\mathbf{SN}(D), \subseteq)$ is a complete lattice. It is, however, not necessarily distributive. The following definition will be used later.

Definition 2.3 *K is a compact stable neighbourhood of $\mathbf{SN}(D)$ if K is a compact Scott open set and a stable neighbourhood. Write $\mathbf{KSN}(D)$ for the set of compact stable neighbourhoods of $\mathbf{SN}(D)$. P is a prime stable neighbourhood of $\mathbf{SN}(D)$ if there exists a finite element d in D such that $P = d\!\uparrow$, where $d\!\uparrow =_{\mathrm{def}} \{ x \in D \mid x \sqsupseteq d \}$.*

If $f : D \to \mathcal{O}$ is a stable function, then clearly $f^{-1}(\top)$ is a stable neighbourhood. On the other hand, suppose U is a stable neighbourhood of D. Then $F(U)$ is a stable function, where

$$F(U)(x) =_{def} \begin{cases} \top & \text{if } x \in U \\ \bot & \text{if } x \in (D \setminus U) \end{cases}$$

However, set inclusion on stable neighbourhoods does not determine the stable order. In $[\mathcal{O} \to_s \mathcal{O}]$, for example, two stable functions $\lambda x.x$ and $\lambda x.\top$ have the property $\forall U \in \mathbf{SN}(\mathcal{O}).\ (\lambda x.x)^{-1}(U) \subseteq (\lambda x.\top)^{-1}(U)$, but we do not have $\lambda x.x \sqsubseteq_s \lambda x.\top$.

Suppose U, V are stable neighbourhoods of D such that $F(U) \sqsubseteq_s F(V)$, where $F : \mathbf{SN}(D) \to [D \to_s \mathcal{O}]$ is defined in the previous paragraph. Then for any x, y in D, $x \sqsubseteq y$ implies $F(U)(x) = F(U)(y) \sqcap F(V)(x)$. Obviously $x \in V$ when $x \in U$, *i.e.*, $U \subseteq V$. Moreover, whenever $x \sqsubseteq y \in U$ but $x \notin U$, it must also be true that $x \notin V$. This means a minimal point of U must also be a minimal point of V.

Definition 2.4 *Let D be a dI-domain. The set of minimal points of $U \in \mathbf{SN}(D)$, write μU, consists of $m \in U$ such that $\forall x \sqsubseteq m.\ x \in U \Longrightarrow x = m$. For $U, V \in \mathbf{SN}(D)$, U minimally less than V, written $U \sqsubseteq_\mu V$, if $\mu U \subseteq \mu V$.*

Clearly $\sqsubseteq_\mu$ is an equivalence relation. $U \sqsubseteq_\mu V$ implies $U \subseteq V$ but not vice versa. Notice that if $U \sqsubseteq_\mu W$ and $U \subseteq V \subseteq W$, then $U \sqsubseteq_\mu V$. $U \sqsubseteq_\mu V$ if and only if $\exists W \in \mathbf{SN}(D).\ V = U \cup W\ \&\ U \cap W = \emptyset$. Every minimal point of a stable neighbourhood is a finite element. The following propositions are now immediate.

Proposition 2.1 *Let D be a dI-domain. We have $[D \to_s \mathcal{O}] \cong \mathbf{SN}(D)$, with the stable order on $[D \to_s \mathcal{O}]$ and $\sqsubseteq_\mu$ on $\mathbf{SN}(D)$, where the isomorphism is given by*

$$f \longmapsto f^{-1}(\top).$$

Proposition 2.2 *$U \in \mathbf{SN}(D)$ implies that there is some $K \subseteq D^0$, a pairwise incompatible set, such that*

$$U = \bigcup \{\, k\uparrow \mid k \in K \,\}.$$

Here D^0 represents the set of finite elements of D.

Suppose $f : D \to E$ is stable. For any $g : E \to \mathcal{O}$, there is a unique stable function $h : D \to \mathcal{O}$ which makes the following diagram commute.

$$\begin{tikzcd} D \arrow[r, "f"] \arrow[dr, "h"] & E \arrow[d, "g"] \\ & \mathcal{O} \end{tikzcd}$$

This implies for any $g : E \to \mathcal{O}$, $f^{-1}(\,g^{-1}(\top)\,) = h^{-1}(\top) \in \mathbf{SN}(D)$, or, for any $U \in \mathbf{SN}(E)$, $f^{-1}(U) \in \mathbf{SN}(D)$, by Proposition 2.1. In general, we have

Theorem 2.1 *Let D, E be dI-domains. $f : D \to E$ is stable if and only if*

$$\forall U \in \mathbf{SN}(E).\ f^{-1}(U) \in \mathbf{SN}(D).$$

For stable functions f, g, $f \sqsubseteq_s g$ if and only if

$$\forall U \in \mathbf{SN}(E).\ f^{-1}(U) \sqsubseteq_\mu g^{-1}(U).$$

Proof The proof for the first part is routine.

We show the sufficiency of the second part, since the necessity of it is implied by Proposition 2.1. For any $x \in D$ and $d \sqsubseteq f(x)$ in E^0, We have $d\uparrow \in \mathbf{SN}(E)$ and $f^{-1}(d\uparrow) \subseteq g^{-1}(d\uparrow)$. Therefore $d \sqsubseteq g(x)$ and hence $f \sqsubseteq g$ under the Scott order. Let $x \sqsubseteq y$ in D, $d' \in E^0$, and $y \sqsupseteq y_0 \in \mu f^{-1}(d'\uparrow)$. By assumption, $y_0 \in \mu g^{-1}(d'\uparrow)$.

$$\begin{aligned} d' \sqsubseteq f(y) \sqcap g(x) \Longrightarrow\ & x \in g^{-1}(d'\uparrow)\ \&\ y \in f^{-1}(d'\uparrow) \\ \Longrightarrow\ & y_0 \uparrow x\ \&\ y_0 \sqcap x \in g^{-1}(d'\uparrow) \\ \Longrightarrow\ & y_0 \sqcap x = y_0\ \ (\text{ as } y_0 \in \mu g^{-1}(d'\uparrow)\ ) \\ \Longrightarrow\ & x \sqsupseteq y_0 \\ \Longrightarrow\ & f(x) \sqsupseteq f(y_0) \sqsupseteq d'. \end{aligned}$$

Therefore $f(x) \sqsupseteq f(y) \sqcap g(x)$. In other words, $f \sqsubseteq_s g$. $\blacksquare$

We remark that if $f \sqsubseteq_s g$, and $x \in \mu g^{-1}(A)$, where A is a stable neighbourhood and $x \in f^{-1}(A)$, then it must be true that $x \in \mu f^{-1}(A)$.

3 Constructions in the Category of DI-Domains

This section studies constructions of stable neighbourhoods in the category of dI-domains with respect to type constructions in this category.

As stated earlier, a dI-domain can be seen as a collection of computations of certain type. The stable neighbourhoods of the dI-domain can be taken as properties about the computations. Constructions on dI-domains can be seen as ways to combine computations together. Suppose x is a computation of type D, having property A, written $x \models A$ and y is a computation of type E, having property B, written $y \models B$. If we combine the computations x of D and y of E together to get a computation $(x \text{ op } y)$ of type $[D \text{ op } E]$ (here op is some domain construction like sum, product, or stable function space), can we deduce some property of $(x \text{ op } y)$ from the facts $x \models A$ and $y \models B$? This question leads to constructions $(A \text{ op } B)$ on properties A and B so that from $x \models A$ and $y \models B$ one deduces $(x \text{ op } y) \models (A \text{ op } B)$.

There can be different ways to combine a stable neighbourhood A of D and a stable neighbourhood B of E together to get a stable neighbourhood $(A \text{ op } B)$ of $[D \text{ op } E]$. But the following are some criteria for good constructions on stable neighbourhoods. In fact most interesting to us are the behaviors of the constructions on compact stable neighbourhoods.

- if A is a stable neighbourhood of D and B is a stable neighbourhood of E, $(A \text{ op } B)$ should be a stable neighbourhood of $[D \text{ op } E]$;
- stable neighbourhoods of the form $(A \text{ op } B)$ should form a subbase of ($[D \text{ op } E]$, $\mathbf{SN}([D \text{ op } E])$);
- if (D, $\mathbf{SN}(D)$) and (E, $\mathbf{SN}(E)$) are sober spaces then so is ($[D \text{ op } E]$, $\mathbf{SN}([D \text{ op } E])$).

The first condition requires that $(A \text{ op } B)$ is well-defined. The second condition states that stable neighbourhoods of the form $(A \text{ op } B)$ are expressive enough. The third condition implies that we can reconstruct the points of the domain from the collection of stable neighbourhoods. All the constructions we are going to introduce do have the above properties. However some of proofs are obvious and we do not always explicitly check them all.

Suppose D and E are dI-domains. Similar to Scott domains, it is easy to get the stable neighbourhoods of $D + E$, $D \times E$ and $D_\perp$ from those of D and E with the requirements for good constructions satisfied. But how can we get the stable neighbourhoods of $[D \to_s E]$ from those of D and E directly?

Let us first have a look at how we dealt with this problem for the Scott topology. If A and B are compact Scott open sets of D and E, respectively, then $A \to B = \{\ h : D \to E \mid A \subseteq h^{-1}(B)\ \}$ is a compact Scott open set of $[D \to E]$. There is another way to look at this. Since A and B are Scott open sets, they correspond to some functions $f_A : D \to \mathcal{O}$, $g_B : E \to \mathcal{O}$. Set inclusion on open sets determines the pointwise order, hence $h \in A \to B$ if and only if $f_A \sqsubseteq g_B \circ h$ (see the diagram below).

$$\begin{tikzcd} D \arrow[r, "h"] \arrow[d, "f_A"] \arrow[dr, "g'"] \arrow[d, phantom, "\sqsubseteq", xshift=4ex] & E \arrow[d, "g_B"] \\ \mathcal{O} & \mathcal{O} \end{tikzcd}$$

This suggests that, for dI-domains, we should use the diagram

$$\begin{tikzcd} D \arrow[r, "h"] \arrow[d, "f_A"] \arrow[dr, "g'"] \arrow[d, phantom, "\sqsubseteq_s", xshift=4ex] & E \arrow[d, "g_B"] \\ \mathcal{O} & \mathcal{O} \end{tikzcd}$$

where the pointwise order is replaced by the stable order. Thus one can define $h \in A \to B$ to mean $f_A \sqsubseteq_s g_B \circ h$. By Theorem 2.1, however, $f_A \sqsubseteq_s g_B \circ h$ if and only if $A \sqsubseteq_\mu h^{-1}(B)$. The following definition is reasonable.

Definition 3.1 *Let D, E be dI-domains, $A \in \mathbf{KSN}(D)$ and $B \in \mathbf{KSN}(E)$. Define $A \to B$ to be the set*

$$\{f \in [D \to_s E] \mid A \sqsubseteq_\mu f^{-1}B\}.$$

Following the view mentioned at the beginning of this section, let us think of a stable function f as a computation of type $[D \to_s E]$ which consumes some information of type D and produces some information of type E (here we can identify the computations of type D and E as data, or information, of type D and E, respectively). What does it mean intuitively for a computation of $[D \to_s E]$ to have a property $A \to B$, where A is a property of type D and B is a property of type E? The properties appropriate for stable functions are those which are determined by a set of incompatible minimal information. We can say that f has property $A \to B$ if f can produce some information with property B from any input information with property A and, moreover, a necessary information of property A is also a necessary information for f to produce some information with property B. We can also say that f has property $A \to B$ if whenever f can produce an output (information) with property B, there is always some minimal input information x for f to do so. If this minimal information x happens to be consistent with property A, then it must also be a minimal information of property A.

We have to show that the construction given in Definition 3.1 does indeed meet the first criteria set at the beginning of the section, that is, the construction is well-defined. For that purpose we have to introduce a characterization of finite elements in the stable function space.

Definition 3.2 *Let $f : D \to E$ be a stable function, where D, E are dI-domains. Define μf to be a set of pairs such that $(a, p) \in \mu f$ if*

$$f(a) \sqsupseteq p \;\&\; \forall a' \sqsubseteq a.\; f(a') \sqsupseteq p \Longrightarrow a = a',$$

where $a \in D^0$, the set of finite elements of D and $p \in E^p$, the set of complete primes of E.

The finite elements in the stable function space are captured by

Theorem 3.1 *Let $f, g \in [D \to_s E]$. $f \sqsubseteq_s g$ if and only if $\mu f \subseteq \mu g$. For $\{ (a_i, b_i) \mid i \in I \} \subseteq D^0 \times E^p$,*

$$\{ (a_i, b_i) \mid i \in I \} = \mu f$$

for some $f \in [D \to_s E]$ if and only if

- $\forall J \subseteq^{fin} I. \{ a_i \mid i \in J \} \uparrow \Longrightarrow \{ b_i \mid i \in J \} \uparrow$,
- $a_i \uparrow a_j \ \& \ (b_i = b_j) \implies (a_i = a_j)$,
- $\forall b \in E^p.\ b_i \sqsupseteq b \implies \exists j.\ b_j = b \ \& \ a_i \sqsupseteq a_j$.

Proof See [10]. ∎

When $\{ (a_i, b_i) \mid i \in I \} \subseteq D^0 \times E^p$ satisfies the three conditions in the above theorem, we call $\{ (a_i, b_i) \mid i \in I \}$ *stable joinable.* Each stable joinable set $\{ (a_i, b_i) \mid i \in I \} \subseteq D^0 \times E^p$ specifies a function

$$\bigsqcup_{i \in I} [a_i, b_i] =_{\text{def}} \lambda x \in D.\ \bigsqcup \{ b_i \mid \exists i \in I.\ a_i \sqsubseteq x \}.$$

They are called *step functions.*

Theorem 3.2 *Assume $A \in \mathbf{KSN}(D)$ and $B \in \mathbf{KSN}(E)$. Then*

$$(A \to B) \in \mathbf{KSN}([D \to_s E]).$$

Proof We show the compactness of $(A \to B)$, and leave it to the reader to check that $(A \to B)$ is indeed a stable neighbourhood. We first prove that stable neighbourhoods of the form $(a\uparrow \to b\uparrow)$ are compact, where $a \in D^0$, $b \in E^0$. Let $Q_a = \{ c \in D \mid c \sqsubseteq a \}$, $P_b = \{ p \in E^1 \mid p \sqsubseteq b \}$, and

$$F_a^b = \{ f \mid f \text{ is a step function } \& \ \mu f \subseteq Q_a \times P_b \ \& \ a \in \mu f^{-1}(b\uparrow) \}.$$

We claim that

$$(a\uparrow \to b\uparrow) = \bigcup_{g \in F_a^b} g\uparrow.$$

It is obvious that

$$(a\!\uparrow \to b\!\uparrow) \supseteq \bigcup_{g \in F_a^b} g\!\uparrow.$$

On the other hand, let $f \in (a\!\uparrow \to b\!\uparrow)$. It can be shown that $[\,a,\, b,\, f\,] \in F_a^b$, where

$$[\,a,\, b,\, f\,] =_{\text{def}} \lambda x.\; b \sqcap f(a \sqcap x).$$

Hence $f \in \bigcup_{g \in F_a^b} g\!\uparrow$ since $f \sqsupseteq_s [\,a,\, b,\, f\,]$. Therefore $(a\!\uparrow \to b\!\uparrow)$ is compact since F_a^b is a finite set.

Write $A = \bigcup_{i \in I}(a_i\!\uparrow)$, $B = \bigcup_{j \in J}(b_j\!\uparrow)$, where I and J are finite and a_i's are pairwise incompatible, b_j's are pairwise incompatible. It is easy to see that

$$\begin{aligned} A \to B \; &= (\textstyle\bigcup_{i \in I} a_i\!\uparrow) \to (\bigcup_{j \in J} b_j\!\uparrow) \\ &= \textstyle\bigcap_{i \in I}(\, a_i\!\uparrow \to \bigcup_{j \in J} b_j\!\uparrow\,) \\ &= \textstyle\bigcap_{i \in I}[\, \bigcup_{j \in J}(\, a_i\!\uparrow \to b_j\!\uparrow\,)\,]. \end{aligned}$$

Hence $A \to B$ is compact. ∎

We have to restrict A and B to compact open sets. Otherwise $(A \to B)$ can be a non-open set, hence not a stable neighbourhood. Consider the dI-domain $\mathbf{N}_\perp$, formed by attaching a bottom element to the set of non-negative integers $\mathbf{N} =^{\text{def}} \{\,0,\, 1,\, \cdots,\, n,\, \cdots\}$. Then set $\mathbf{N}$ is clearly a stable neighbourhood of $\mathbf{N}_\perp$. The set, $\{\, f : \mathbf{N}_\perp \to \mathcal{O} \mid \mathbf{N} \subseteq_\mu f^{-1}(\top)\,\}$, is not open since one can easily produce a chain of stable functions whose limit is in this set but not any of the finite approximations.

We remark that unlike in the case of Scott's continuous functions, $a\!\uparrow \to b\!\uparrow$ need not be a prime stable neighbourhood when a and b are finite elements.

The theorem given below implies the second and the third properties required for a construction of the stable neighbourhoods set at the beginning of the section. In particular, this theorem says that for a stable joinable set in the function space, if we take the intersection of the stable neighbourhoods $(\,a_i\!\uparrow \to b_i\!\uparrow\,)$, $i \in I$, we get a prime stable neighbourhood consisting of all the stable functions which dominate the stable step function $\bigsqcup_{i \in I}[\,a_i,\, b_i\,]$ under the stable order.

Theorem 3.3 *Let* $\{ (a_i,\ b_i) \mid i \in I \}$ *be a stable joinable set. Then*

$$\bigcap_{i \in I} (a_i{\uparrow} \rightarrow b_i{\uparrow}) = \Big[\bigsqcup_{i \in I} [\, a_i, b_i \,] \Big]{\uparrow}\ .$$

Proof We know from Theorem 3.1 that

$$\bigsqcup_{i \in I} [\, a_i, b_i \,] \in \bigcap_{i \in I} (a_i{\uparrow} \rightarrow b_i{\uparrow}).$$

It is enough to show that $\bigsqcup_{i \in I} [\, a_i, b_i \,]$ is less than or equal to any other stable function in $\bigcap_{i \in I} (a_i{\uparrow} \rightarrow b_i{\uparrow})$. Let g be a stable function in $\bigcap_{i \in I} (a_i{\uparrow} \rightarrow b_i{\uparrow})$. For any $i \in I$, $g(a_i) \sqsupseteq b_i$. Therefore for any x in D

$$\begin{aligned} \bigsqcup \{ b_k \mid a_k \sqsubseteq x \} &\sqsubseteq \bigsqcup \{ g(a_k) \mid a_k \sqsubseteq x \} \\ &\sqsubseteq g(x), \end{aligned}$$

i.e.,

$$\bigsqcup_{i \in I} [\, a_i, b_i \,](x) \sqsubseteq g(x).$$

Suppose $x, y \in D$ and $x \sqsubseteq y$. Let $p \sqsubseteq \bigsqcup \{ b_j \sqcap g(x) \mid a_j \sqsubseteq y \}$, where p is a complete prime. $p \sqsubseteq b_j \sqcap g(x)$ for some j. Therefore, there exists s such that $p = b_s$ and $a_j \sqsupseteq a_s$. $g(a_s \sqcap x) = g(a_s) \sqcap g(x) \sqsupseteq b_s$. This implies, as $g \in (a_s{\uparrow} \rightarrow b_s{\uparrow})$, $a_s \sqcap x = a_s$, or $a_s \sqsubseteq x$. Hence $p \sqsubseteq \bigsqcup \{ b_i \mid a_i \sqsubseteq x \}$. By the prime algebraicity of E,

$$\begin{aligned} g(x) \sqcap \bigsqcup \{ b_i \mid a_i \sqsubseteq y \} &= \bigsqcup \{ b_j \sqcap g(x) \mid a_j \sqsubseteq y \} \\ &\sqsubseteq \bigsqcup \{ b_i \mid a_i \sqsubseteq x \}. \end{aligned}$$

Now it is easy to see that

$$\bigsqcup_{i \in I} [\, a_i, b_i \,] \sqsubseteq_s g.$$

∎

From this theorem we can also see that it is possible to get all the compact stable neighbourhoods of $[D \rightarrow_s E]$ by finite union and intersection of stable neighbourhoods of the form $A \rightarrow B$, where A, B are compact stable neighbourhoods of D and E, respectively. Moreover we can get every prime stable neighbourhood in the stable function space in this way. Therefore, we have, for f, $g \in [D \to_s E]$, $f \sqsubseteq_s g$ if and only if $f \in (A \to B)$ implies $g \in (A \to B)$ for all $A \in \mathbf{KSN}(D)$, $B \in \mathbf{KSN}(E)$.

Some proof rules now follow. We let the rules take care of the types themselves.

Proposition 3.1 *Let A, B, C, D be stable neighbourhoods and a a finite element. Then with appropriate types we have*

$$A \cap B = \emptyset \Longrightarrow (A \cup B) \to C = (A \to C) \cap (B \to C),$$
$$A \cap B = \emptyset \Longrightarrow a\!\uparrow\to (A \cup B) = (a\!\uparrow\to A) \cup (a\!\uparrow\to B),$$
$$(A \to C) \cap (B \to D) \subseteq (A \cap B) \to (C \cap D).$$

Proof Only the last inequality needs verification. Let $f \in (A \to C) \cap (B \to D)$. We have $A \sqsubseteq_\mu f^{-1}C$ and $B \sqsubseteq_\mu f^{-1}D$. One can check that $A \cap B \sqsubseteq_\mu (f^{-1}C) \cap (f^{-1}D)$. But $(f^{-1}C) \cap (f^{-1}D) = f^{-1}(C \cap D)$. Therefore the desired inequality follows. ∎

Note however, the rule

$$A' \subseteq A \;\&\; B \subseteq B' \Longrightarrow (A \to B) \subseteq (A' \to B')$$

which is sound for Scott continuous functions and appeared in Hoare logic, is no longer valid for stable functions. That is because with stable functions, $f \in A \to B$ if and only if $A \sqsubseteq_\mu f^{-1}B$, *not* simply $A \subseteq f^{-1}B$!

Fortunately, we do have a rule which would help.

Proposition 3.2 *Let $a \in D^0$ and b, $c \in E^0$, where D, E are dI-domains.*

$$c \sqsubseteq b \Longrightarrow (a\!\uparrow\to b\!\uparrow) \subseteq \bigcup_{a' \sqsubseteq a} (a'\!\uparrow\to c\!\uparrow).$$

Note that if $a' \neq a''$ and $a' \uparrow a''$ then $(a'\!\uparrow\to c\!\uparrow) \cap (a''\!\uparrow\to c\!\uparrow) = \emptyset$. Hence $\bigcup_{a' \sqsubseteq a}(a'\!\uparrow\to c\!\uparrow)$ is a stable neighbourhood of $[D \to_s E]$ since it is a disjoint union.

Proof Suppose $f \in (a\!\uparrow\to b\!\uparrow)$. Then $a \in \mu f^{-1} b\!\uparrow$. We have $f(a) \sqsupseteq c$. Let $a'' = \bigsqcap \{\, x \mid x \sqsubseteq a \;\&\; f(x) \sqsupseteq c \,\}$. Clearly $a'' \sqsubseteq a$ and $a'' \in \mu f^{-1} c\!\uparrow$. Hence $f \in \bigcup_{a' \sqsubseteq a}(a'\!\uparrow\to c\!\uparrow)$. ∎

4 Constructions in the Category of Coherent Spaces

Since coherent spaces are special kinds of dI-domains, our results in the previous section apply. However, the category of coherent spaces has some type constructions of its own, which we deal with now. Of course, for each construction of stable neighbourhoods, we still have to guarantee that the requirements set at the beginning of the previous section are met.

A coherent space is a set of subsets $\mathcal{F}$ of a countable set E which satisfies

- $\forall x \in \mathcal{F}.\, y \subseteq x \Longrightarrow y \in \mathcal{F}$
- $(\, X \subseteq \mathcal{F} \;\&\; \forall x, y \in X.\; x \uparrow y \,) \Longrightarrow \bigcup X \in \mathcal{F}$

Sometimes we call members of $\mathcal{F}$ *configurations*. Let $\mathcal{F}$ be a coherent space. Then $(\, \mathcal{F}, \subseteq \,)$ is a dI-domain with $\{\, \{\, a \,\} \mid \{\, a \,\} \in \mathcal{F} \,\}$ the set of its complete primes.

There are two well-known categories of coherent spaces. One is the cartesian closed category of coherent spaces with stable functions, and the other is the monoidal closed category of coherent spaces with linear, stable functions. As discovered by Girard ([6]), however, the stable function space can be decomposed into two more basic constructions: the linear function space, and the exponential. The exponential actually gives an adjunction between the two categories. Therefore it is enough to deal with the monoidal closed category in detail.

We briefly review some useful constructions in the category of coherent spaces and linear maps.

Tensor Product Let $\mathcal{F}_0, \mathcal{F}_1$ be coherent spaces. Their *tensor product*, $\mathcal{F}_0 \otimes \mathcal{F}_1$, is a family of subsets which satisfies

$$x \in \mathcal{F}_0 \otimes \mathcal{F}_1 \iff \begin{array}{l} \bullet\; x \subseteq \bigcup \mathcal{F}_0 \times \bigcup \mathcal{F}_1 \;\& \\ \bullet\; \pi_0 x \in \mathcal{F}_0 \;\&\; \pi_1 x \in \mathcal{F}_1 \end{array}$$

Linear Function Space Let $\mathcal{F}_0, \mathcal{F}_1$ be coherent spaces. Their *linear function* *space*, $\mathcal{F}_0 \multimap \mathcal{F}_1$, is a family of subsets which satisfies

$$x \in \mathcal{F}_0 \multimap \mathcal{F}_1 \iff \begin{array}{l} \bullet\ x \subseteq \bigcup \mathcal{F}_0 \times \bigcup \mathcal{F}_1\ \& \\ \bullet\ \forall y \subseteq x.\ \pi_0 y \in \mathcal{F}_0 \Longrightarrow \pi_1 y \in \mathcal{F}_1\ \& \\ \bullet\ [\{\, e_0,\, e_0' \,\} \in \mathcal{F}_0\ \&\ (\, e_0,\, e_1 \,),\ (\, e_0',\, e_1 \,) \in x] \Longrightarrow e_0 = e_0' \end{array}$$

Any element x in $\mathcal{F}_0 \multimap \mathcal{F}_1$ determines a linear stable function $Pt\, x$, where

$$Pt\, x\,(x_0) = \{\, e_1 \in \bigcup \mathcal{F}_1 \mid \exists e_0 \in x_0.\, (\, e_0,\, e_1 \,) \in x \,\}.$$

Linear function space is coherent, and is isomorphic to the linear functions from $\mathcal{F}_0$ to $\mathcal{F}_1$.

With respect to the tensor product and linear function space introduced above, we have the following isomorphism:

$$[\mathcal{F}_0 \otimes \mathcal{F}_1 \multimap \mathcal{F}_2] \cong [\mathcal{F}_0 \multimap [\mathcal{F}_1 \multimap \mathcal{F}_2]].$$

Exponential Let $\mathcal{F}$ be a coherent family. Its *exponential*, $!\mathcal{F}$, is a family of subsets which satisfies

$$x \in !\mathcal{F} \iff x \subseteq \{\, a \mid a \in \mathcal{F}\ \&\ \mid a \mid < \infty \,\}\ \&\ \bigcup x \in \mathcal{F}$$

It is obvious that if $\mathcal{F}$ is a coherent family then $!\mathcal{F}$ is a coherent family again. The requirement that x consist of the *finite* elements of $\mathcal{F}$ ensures that there are countably many tokens in $!\mathcal{F}$.

Now we introduce constructions on the disjunctive spaces related to the tensor product, the linear function space, and the exponential. We show that all these constructions preserve compactness. We also give rules which indicate how those constructions interact with unions and intersections.

Theorem 4.1 *Let $\mathcal{F}_0$, $\mathcal{F}_1$ be coherent spaces and $A \in \mathbf{KSN}(\mathcal{F}_0)$, $B \in \mathbf{KSN}(\mathcal{F}_1)$. Then $A \otimes B \in \mathbf{KSN}(\,\mathcal{F}_0 \otimes \mathcal{F}_1\,)$, where*

$$A \otimes B =_{def} \{\, x \in \mathcal{F}_0 \otimes \mathcal{F}_1 \mid \exists x_0 \in A,\, x_1 \in B.\ x_0 \times x_1 \subseteq^{fin} x \,\},$$

$A \multimap B \in \mathbf{KSN}(\,\mathcal{F}_0 \multimap \mathcal{F}_1\,)$, *where*

$$A \multimap B =_{def} \{\; x \in \mathcal{F}_0 \multimap \mathcal{F}_1 \mid A \sqsubseteq_\mu (Pt\,x)^{-1} B \;\},$$

and $!A \in \mathbf{KSN}(\,!\mathcal{F})$, *where* $!A =_{def} \{\; x \in !\mathcal{F} \mid x \cap \mu A \neq \emptyset \;\}$.

Proof Take the first conclusion as an example. Clearly $A \otimes B$ is a Scott open set. Now assume $x,\, y \in A \otimes B$ and $x \uparrow y$. Then there exist $x_0,\, x_1 \in A$, $y_0,\, y_1 \in B$ such that $x_0 \times y_0 \subseteq^{fin} x$, $x_1 \times y_1 \subseteq^{fin} y$. We have $x_0 \uparrow x_1$ and $y_0 \uparrow y_1$. Therefore $x_0 \cap x_1 \in A$, $y_0 \cap y_1 \in B$. Also, $(x_0 \cap x_1) \times (y_0 \cap y_1) \subseteq^{fin} x \cap y$. Hence $x \cap y \in A \otimes B$.

To check compactness we first show that $(a\!\uparrow \otimes\, b\!\uparrow)$ is compact, where a, b are finite configurations of $\mathcal{F}_0$ and $\mathcal{F}_1$, respectively. Clearly $a \times b \in (a\!\uparrow \otimes\, b\!\uparrow)$. On the other hand, suppose $u \in (a\!\uparrow \otimes\, b\!\uparrow)$. Then by definition, there exist a', b', $a \subseteq a' \in \mathcal{F}_0$, $b \subseteq b' \in \mathcal{F}_1$, such that $a' \times b' \subseteq^{fin} u$. Hence $a \times b \subseteq^{fin} u$. We have shown that

$$(a\!\uparrow \otimes\, b\!\uparrow) = (a \times b)\!\uparrow,$$

therefore $(a\!\uparrow \otimes\, b\!\uparrow)$ is compact. By Proposition 2.2 and the fact that $A \otimes (B \cup C) = (A \otimes B) \cup (A \otimes C)$ when $B \cap C = \emptyset$, we deduce that any $A \otimes B$ is compact. ∎

From the proof we can see that

$$\widehat{e_0} \otimes \widehat{e_1} = \{\; x \in \mathcal{F}_0 \otimes \mathcal{F}_1 \mid (e_0,\, e_1) \in x \;\},$$

where $\widehat{e_0} = \{\; x_0 \in \mathcal{F}_0 \mid e_0 \in x_0 \;\}$ and $\widehat{e_1} = \{\; x_1 \in \mathcal{F}_1 \mid e_1 \in x_1 \;\}$. It is clear that by using finite union and finite intersection we can get all compact stable neighbourhood of $\mathcal{F}_0 \otimes \mathcal{F}_1$ out of compact neighbourhoods of the form $A \otimes B$. It is also clear that for $x,\, y \in \mathcal{F}_0 \otimes \mathcal{F}_1$, $x \subseteq y$ if and only if $x \in (A \otimes B)$ implies $y \in (A \otimes B)$ for all $A \in \mathbf{SN}(\mathcal{F}_0)$, $B \in \mathbf{SN}(\mathcal{F}_1)$.

We remark that

$$(\widehat{e_0} \multimap \widehat{e_1}) = \{\; x \in \mathcal{F}_0 \multimap \mathcal{F}_1 \mid (e_0,\, e_1) \in x \;\},$$

where $\widehat{e_0} = \{\; x_0 \in \mathcal{F}_0 \mid e_0 \in x_0 \;\}$ and $\widehat{e_1} = \{\; x_1 \in \mathcal{F}_1 \mid e_1 \in x_1 \;\}$. In fact, let $x \in \mathcal{F}_0 \multimap \mathcal{F}_1$ and $(e_0,\, e_1) \in x$. Then $\{\, e_1 \,\} \subseteq Pt\,x\;\{\, e_0 \,\}$, and $\{\, e_0 \,\} \in \mu(Pt\,x)^{-1}\widehat{e_1}$.

So $x \in (\widehat{e_0} \multimap \widehat{e_1})$. Suppose, on the other hand, that $x \in (\widehat{e_0} \multimap \widehat{e_1})$. We have $\{ e_0 \} \in \mu(Pt\, x)^{-1}\widehat{e_1}$. Therefore, $Pt\, x \, \{ e_0 \} \supseteq \{ e_1 \}$ and $(e_0, e_1) \in x$.

By this observation, each compact stable neighbourhood of $\mathcal{F}_0 \multimap \mathcal{F}_1$ can be constructed out of $A \multimap B$ by using finite union and finite intersection. Also, for $x, y \in \mathcal{F}_0 \multimap \mathcal{F}_1$, $x \subseteq y$ if and only if $x \in (A \multimap B)$ implies $y \in (A \multimap B)$ for all $A \in \mathbf{SN}(\mathcal{F}_0)$, $B \in \mathbf{SN}(\mathcal{F}_1)$.

Therefore, the constructions given in Theorem 4.1 do meet the requirements set at the beginning of Section 3.

To suggest some proof rules, we present some results on the properties of the constructions given in Theorem 4.1 in the rest of the section. The not-so-hard proofs are left to the reader.

Proposition 4.1 *Suppose $A_1, A_2 \in \mathbf{KSN}(\mathcal{F}_0)$, $B_1, B_2 \in \mathbf{KSN}(\mathcal{F}_1)$, where $\mathcal{F}_0$, $\mathcal{F}_1$ are coherent spaces. Then*

$$\begin{aligned} &(A_1 \cap A_2) \otimes (B_1 \cap B_2) \\ &= (A_1 \otimes B_1) \cap (A_2 \otimes B_1) \cap (A_1 \otimes B_2) \cap (A_2 \otimes B_2). \end{aligned}$$

Similar to Proposition 3.1, there is

Proposition 4.2 *Let A, B, C, D be stable neighbourhoods and a a finite configuration. Then with appropriate types we have*

$$\begin{aligned} &A \cap B = \emptyset \Longrightarrow (A \cup B) \multimap C = (A \multimap C) \cap (B \multimap C) \\ &A \cap B = \emptyset \Longrightarrow a{\uparrow} \multimap (A \cup B) = (a{\uparrow} \multimap A) \cup (a{\uparrow} \multimap B) \\ &(A \multimap C) \cap (B \multimap D) \subseteq (A \cap B) \multimap (C \cap D) \end{aligned}$$

Note the rule $A' \subseteq A \,\&\, B \subseteq B' \Longrightarrow (A \multimap B) \subseteq (A' \multimap B')$ is not valid.

Proposition 4.3 *Let $x \in \mathcal{F}_0$ and $y \in \mathcal{F}_1$ be finite configurations and $\mathcal{F}_0$, $\mathcal{F}_1$ coherent spaces. We have $e' \in y \Longrightarrow (x{\uparrow} \multimap y{\uparrow}) \subseteq \bigcup_{e \in x}(\widehat{e} \multimap \widehat{e'})$.*

For the exponential, we have

Proposition 4.4 *Let A, $B \in \mathbf{SN}(\mathcal{F})$ and $A \cap B = \emptyset$. Then*

$$!(A \cup B) = (!A) \cup (!B).$$

Of course Definition 3.1 of the previous section specializes down to coherent spaces. For coherent spaces $\mathcal{F}_0$ and $\mathcal{F}_1$ with $A \in \mathbf{KSN}(\mathcal{F}_0)$, $B \in \mathbf{KSN}(\mathcal{F}_1)$, we define

$$A \to B = \{\, x \in \mathcal{F}_0 \to \mathcal{F}_1 \mid A \sqsubseteq_\mu (Pt\, x)^{-1} B \,\}.$$

As a corollary of Theorem 3.2, we have $A \to B \in \mathbf{KSN}(\mathcal{F}_0 \to \mathcal{F}_1)$. It is well-known that there is an isomorphism $\ell : [\mathcal{F}_0 \to_s \mathcal{F}_1] \to [!\mathcal{F}_0 \multimap \mathcal{F}_1]$. Is it also true, under the isomorphism, that $A \to B \sim (!A) \multimap B$, as one may expect? Of course. We have the following proposition, whose proof is omitted.

Proposition 4.5 $x \in A \to B$ if and only if $\ell x \in (!A) \multimap B$.

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